Chapter Seven: Systems of Equations and Inequalities

7.1 Linear and Nonlinear Systems of Equations

Definition: A system of equations consists of two or more equations considered together. The solution to a system is the single point that is a solution to every equation in the system. Sometimes there is no solution, when they do not share a common point, and other times the lines may be the same and therefore every point is a solution.

We have two general ways of solving systems: substitution and elimination.

Substitution Method:

1. Solve one of the equations for one variable in terms of the other.

2. Substitute the expression found in step 1 into the other equation to obtain an equation in one variable.

3. Solve the equation obtained in step 2.

4. Back-substitute the value obtained in step 3 into the expression obtained in step 1 to find the value of the other variable.

5. Check that the solution satisfies each of the original equations.

Examples: Solve by substitution.

1. $\begin{cases} x+4y=3 \\ 2x-7y=-24 \end{cases}$

① Solve eq1 for x:

$x+4y=3$

$x=3-4y$

② Substitute:

$2(3-4y)-7y=-24$

③ Solve

$6-8y-7y=-24$

$6-15y=-24$

$-15y=-30$

$y=2$

④ Back substitute

$x=3-4(2)=3-8=-5$

$\boxed{\text{Solution: } (-5,2)}$

2. $\begin{cases} 6x-3y-4=0 \\ x+2y-4=0 \end{cases}$

① Solve eq2 for x:

$x=4-2y$

② Substitute

$6(4-2y)-3y-4=0$

③ Solve

$24-12y-3y-4=0$

$20-15y=0$

$20=15y$

$\frac{20}{15}=y=\frac{4}{3}$

④ Back-sub

$x=4-2\left(\frac{4}{3}\right)=4-\frac{8}{3}$

$=\frac{4}{1}\cdot\frac{3}{3}-\frac{8}{3}$

$=\frac{12}{3}-\frac{8}{3}=\frac{4}{3}$

$\boxed{\text{Solution } \left(\frac{4}{3},\frac{4}{3}\right)}$

3. $\begin{cases} x-2y=0 \\ 3x-y^2=0 \end{cases}$

① $x=2y$

② $3(2y)-y^2=0$

③ $6y-y^2=0$

$y(6-y)=0$

$y=0$ or $6=y$

④ for $y=0$

$x=2(0)=0$

for $y=6$

$x=2(6)=12$

$\boxed{\text{Solutions: } (0,0) \quad (12,6)}$

4. $\begin{cases} x-y-1=0 \\ x^2+y^2-4x=0 \end{cases}$

① $x-1=y$

② $x^2+(x-1)^2-4x=0$

③ $x^2+(x-1)(x-1)-4x=0$

$x^2+x^2 \quad x-x+1-4x=0$

$2x^2-6x+1=0$

$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(2)(1)}}{2(2)}$

$x=\dfrac{6\pm\sqrt{36-8}}{4}$

$x=\dfrac{6\pm\sqrt{28}}{4}$

$\sqrt{28}=\sqrt{4\cdot7}=2\sqrt{7}$

$x=\dfrac{6\pm2\sqrt{7}}{4}$

$x=\dfrac{2(3\pm\sqrt{7})}{2(2)}$

$x=\dfrac{3\pm\sqrt{7}}{2}$

$x=\dfrac{3+\sqrt{7}}{2}, \dfrac{3-\sqrt{7}}{2}$

④ for $x=\dfrac{3+\sqrt{7}}{2}$

$y=\dfrac{3+\sqrt{7}}{2}-1=\dfrac{1+\sqrt{7}}{2}$

$\left(\frac{3}{2}-1=\frac{1}{2}\right)$

for $x=\dfrac{3-\sqrt{7}}{2}$

$y=\dfrac{3-\sqrt{7}}{2}-1=\dfrac{1-\sqrt{7}}{2}$

Solutions:

$\left(\dfrac{3+\sqrt{7}}{2}, \dfrac{1+\sqrt{7}}{2}\right)$

$\left(\dfrac{3-\sqrt{7}}{2}, \dfrac{1-\sqrt{7}}{2}\right)$

5. You try it: $\begin{cases} 2x+y=3 \\ 4x-5y=-2 \end{cases}$

Example: A small software company invests \$25,000 to produce a software package that will sell for \$69.95. Each unit can be produced for \$45.25.

Costs 45.25 per unit = $45.25x$
25,000 fixed
$C(x) = 45.25x + 25{,}000$

Revenue 69.95 per unit
$R(x) = 69.95x$

Let $x$ be number of units

a) How many units must be sold to break even? → $R = C$

$$69.95x = 45.25x + 25{,}000$$
$$-45.25x \qquad -45.25x$$
$$24.7x = 25{,}000$$

$$x = \frac{25{,}000}{24.7} = 1012.145\ldots$$

About 1012 units to break even.

b) How many units must be sold to make a profit of \$100,000?

→ Profit = Revenue − Cost
$P(x) = R(x) - C(x)$

$$P(x) = 69.95x - (45.25x + 25{,}000)$$
$$P(x) = 69.95x - 45.25x - 25{,}000$$
$$P(x) = 24.70x - 25{,}000$$
$$100{,}000 = 24.7x - 25{,}000$$
$$125{,}000 = 24.7x$$

$$\frac{125{,}000}{24.7} = x \approx 5060.728\ldots$$

To make a profit of \$100,000, you would need to make and sell 5061 units.

Example: You are offered two jobs selling dental supplies. One company offers a straight commission of 6% of sales. The other company offers a salary of \$500 per week plus 3% of sales. How much would you have to sell in a week in order to make the straight commission offer better?

Straight Commission = $C(x) = .06x$ if $x$ = amount of sales
Salary = $S(x) = .03x + 500$

Straight commission better implies $C(x) > S(x)$

$$.06x > .03x + 500$$
$$-.03x \qquad -.03x$$
$$.03x > 500$$
$$\frac{.03x}{.03} > \frac{500}{.03}$$
$$x > 16{,}666.67$$

You must sell over \$16,666.67 in order for straight commission to be better.