7.3 Multivariable Linear Systems

The method of elimination can be applied to any system with any number of variables. However, after you pass two variables you want to make sure that you have a system to keep track of your steps and to be sure to keep making progress. With elementary operations we can change this

$$\begin{cases} x-2y+3z=9 \\ -x+3y \qquad =-4 \\ 2x-5y+5z=17 \end{cases}$$

into a similar system that is much more ordered

$$\begin{cases} x-2y+3z=9 \\ \qquad y+3z=5 \\ \qquad\qquad z=2 \end{cases}.$$

This second system is said to be in row-echelon form, which means that it has a "stair-step" pattern with leading coefficients of 1. With this second system we can use back substitution to find the value of each variable.

Examples: Use back-substitution to solve the system of linear equations.

1. $\begin{cases} 4x-3y-2z=21 \\ \qquad 6y-5z=-8 \\ \qquad\qquad z=-2 \end{cases}$

$z=-2$

$6y-5(-2)=-8$
$6y+10=-8$
$6y=-18$
$y=-3$

$4x-3(-3)-2(-2)=21$
$4x+9+4=21$
$4x+13=21$
$4x=8$
$x=2$

Solution: $(2,-3,-2)$
(x,y,z)

2. $\begin{cases} x-y+2z=22 \\ \qquad 3y-8z=-9 \\ \qquad\qquad z=-3 \end{cases}$

$z=-3$

$3y-8(-3)=-9$
$3y+24=-9$
$3y=-33$
$y=-11$

$x-(-11)+2(-3)=22$
$x+11-6=22$
$x+5=22$
$x=17$

Solution: $(17,-11,-3)$

Operations That Produce Equivalent Systems – Each of the following row operations on a system of linear equations produces an equivalent system of linear equations:

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to another equation to replace the latter equation.

Solving systems of equations using these operations is called Gaussian Elimination (or Gauss-Jordan elimination) after mathematician Carl Freidrich Gauss.

Examples: Solve the system of linear equations and check any solution algebraically.

1. $\begin{cases} x+y+z=5 & \text{EQ1} \\ x-2y+4z=13 & \text{EQ2} \\ 3y+4z=13 & \text{EQ3} \end{cases}$

First: make top equation have 1x then eliminate all x's below:

$$\begin{array}{ll} \text{EQ1} & x+y+z=5 \\ -\text{EQ2} & -x+2y-4z=-13 \\ \hline \text{New EQ2} & 3y-3z=-8 \end{array}$$

$$\begin{cases} x+y+z=5 \\ 3y-3z=-8 \\ 3y+4z=13 \end{cases}$$

Second: eliminate all y's below second equation (1y in EQ2 is optional)

$$\begin{array}{ll} \text{EQ2} & 3y-3z=-8 \\ -\text{EQ3} & -3y-4z=-13 \\ \hline \text{New EQ3} & -7z=-21 \end{array}$$

Solve: $z=3$

$$\begin{cases} x+y+z=5 \\ 3y-3z=-8 \\ z=3 \end{cases}$$

Back-substitute to get

$\left(\frac{5}{3}, \frac{1}{3}, 3\right)$

2. $\begin{cases} 2x+4y+z=1 \\ x-2y-3z=2 \\ x+y-z=-1 \end{cases}$

switch EQ1 + EQ3 to make a 1x

$\begin{cases} x+y-z=-1 \\ x-2y-3z=2 \\ 2x+4y+z=1 \end{cases}$

eliminate x's below first equation

EQ1 − EQ2 → New EQ2:
$$\begin{array}{r} x+y-z=-1 \\ -x+2y+3z=2 \\ \hline 3y+2z=-3 \end{array}$$

−2EQ1 + EQ3 → New EQ3:
$$\begin{array}{r} -2x-2y+2z=2 \\ 2x+4y+z=1 \\ \hline 2y+3z=3 \end{array}$$

$\begin{cases} x+y-z=-1 \\ 3y+2z=-3 \\ 2y+3z=3 \end{cases}$

eliminate y's below second equation

2EQ2 − 3EQ3 → New EQ3:
$$\begin{array}{r} 6y+4z=-6 \\ -6y-9z=-9 \\ \hline -5z=-15 \end{array}$$

solve $z=3$

$\begin{cases} x+y-z=-1 \\ 3y+2z=-3 \\ z=3 \end{cases}$

Back-substitute to get $(5,-3,3)$

3. $\begin{cases} 2x+4y-z=7 \\ 2x-4y+2z=-6 \\ x+4y+z=0 \end{cases}$

switch EQ1 + EQ3

$\begin{cases} x+4y+z=0 \\ 2x-4y+2z=-6 \\ 2x+4y-z=7 \end{cases}$

eliminate x's.

−2EQ1 + EQ2 → New EQ2:
$$\begin{array}{r} -2x-8y-2z=0 \\ 2x-4y+2z=-6 \\ \hline -12y=-6 \end{array}$$

solve $y=\frac{1}{2}$

−2EQ1 + EQ3 → New EQ3:
$$\begin{array}{r} -2x-8y-2z=0 \\ 2x+4y-z=7 \\ \hline -4y-3z=7 \end{array}$$

$\begin{cases} x+4y+z=0 \\ y=\frac{1}{2} \\ -4y-3z=7 \end{cases}$

we could solve from here. If you see it, do it

eliminate y's

4EQ2 + EQ3 → New EQ3:
$$\begin{array}{r} 4y=2 \\ -4y-3z=7 \\ \hline -3z=9 \end{array}$$

solve $z=-3$

$\begin{cases} x+4y+z=0 \\ y=\frac{1}{2} \\ z=-3 \end{cases}$

Back-substitute to find x and get $(1,\frac{1}{2},-3)$

4. $\begin{cases} 2x-y-z=0 \\ -2x+6y+4z=2 \end{cases}$

add the equations to get new EQ2

$$\begin{cases} 2x-y-z=0 \\ 5y+3z=2 \end{cases}$$

when there are more variables than equations you will have a dependent system. To solve for WebAssign, let $z=a$ then solve for both y and x:

$$5y+3a=2$$
$$5y=2-3a$$
$$y=\frac{2-3a}{5}$$

$$2x-\frac{2-3a}{5}-a=0$$
$$2x=\frac{2-3a}{5}+\frac{a}{1}\cdot\frac{5}{5}$$
$$2x=\frac{2+2a}{5}$$
$$x=\frac{1+a}{5}$$

Solution: $\left(\frac{1+a}{5}, \frac{2-3a}{5}, a\right)$

Example: In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from a combination of touchdowns (6 points each), extra-point kicks (1 point each) and field goals (3 points each). The numbers of touchdowns and extra-point kicks were equal. There were six times as many touchdowns as field goals. Find the numbers of touchdowns, extra-point kicks, and field goals scored.

total points = 35+10 = 45

Let x = number of touchdowns
y = number of extra-points
z = number of field goals

$6x+1y+3z=45$ total points

$x=y$ becomes $x-y=0$

$x=6z$ becomes $x-6z=0$

already more touchdowns so multiply field goals by 6 to make them equal

$$\begin{cases} 6x+y+3z=45 \\ x-y=0 \\ x-6z=0 \end{cases}$$

Switch EQ1 + EQ2

$$\begin{cases} x-y=0 \\ 6x+y+3z=45 \\ x-6z=0 \end{cases}$$

eliminate x's:

-6EQ1: $-6x+6y=0$
$+$ EQ2: $6x+y+3z=45$
New EQ2: $7y+3z=45$

EQ1: $x-y=0$
$-$ EQ3: $-x+6z=0$
New EQ3: $-y+6z=0$

$$\begin{cases} x-y=0 \\ 7y+3z=45 \\ -y+6z=0 \end{cases}$$

eliminate y's

EQ2: $7y+3z=45$
$+7$EQ3: $-7y+42z=0$
new EQ3: $45z=45$

Solve $z=1$

$$\begin{cases} x-y=0 \\ 7y+3z=45 \\ z=1 \end{cases}$$

Back substitute:

$7y+3(1)=45$
$7y=42$
$y=6$ extra points and
$x=6$ touchdowns
with $z=1$ field goals.