7.4 Partial Fractions

Finding a partial fraction decomposition is the opposite operation of finding a common denominator. We are tearing a rational expression apart into its component pieces.

Decomposition of N(x)/D(x) into Partial Fractions:

1. Divide if improper: If N(x)/D(x) is an improper fraction, divide the denominator into the numerator to obtain a polynomial plus a proper fraction.

2. Factor the denominator: Completely factor the denominator into factors of the form $(px+q)^m$ and $(ax^2+bx+c)^n$ where the quadratic is irreducible.

3. Linear factors: For each factor of the linear factors, the partial fraction decomposition must include the following sum of *m* fractions

$$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\dots+\frac{A_m}{(px+q)^m}$$

4. Quadratic Factors: For each quadratic factor, the partial fraction decomposition must include the following sum of *n* fractions

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\dots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Examples: Write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

1. $\frac{x-2}{x^2+4x+3}$ factor denominator $\frac{x-2}{(x+3)(x+1)}$

Linear factors become $\frac{A}{x+3}+\frac{B}{x+1}$

1) Order does not matter

2) Linear denominator has constant numerator. Always

2. $\frac{x^2-3x+2}{4x^3+11x^2} = \frac{x^2-3x+2}{x^2(4x+11)}$

repeated linear factor $x^2 = x \cdot x$

so x in 2 powers

$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{4x+11}$

3. $\dfrac{6x+5}{(x+2)^4}$ already factored $\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}+\dfrac{D}{(x+2)^4}$

need x+2 in 4 powers

4. $\dfrac{x+3}{\underbrace{(x-1)}_{\text{linear}}\underbrace{(x^2+x+1)}_{\text{quad}}}$ already factored as x^2+x+1 is irreducible (has complex zeros)

form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$

1) order still doesn't matter

2) A quadratic factor has a linear numerator. Always

5. $\dfrac{17}{x(x^2-5x-4)^2}$ already factored

twice

form $\dfrac{A}{x}+\dfrac{Bx+C}{x^2-5x-4}+\dfrac{Dx+E}{(x^2-5x-4)^2}$

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the zeros of the distinct linear factors into the basic equation.

2. For repeated linear factors, use the coefficients determined in step 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.

2. Collect terms according to powers of x.

3. Equate the coefficients of like terms to obtain equations involving A, B, C, and so on.

4. Use a system of linear equations to solve for A, B, C, …

Examples: Write the partial fraction decomposition of the rational expression.

1. $\frac{3}{x^2-3x} = \frac{3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$ multiply by common denominator $x(x-3)$

$\cancel{x(x-3)}\frac{3}{\cancel{x(x-3)}} = \cancel{x}(x-3)\frac{A}{\cancel{x}} + x\cancel{(x-3)}\frac{B}{\cancel{x-3}}$ gives $3 = A(x-3) + Bx$

using zeros of original denominator:

when $x=0$: $3 = A(0-3) + B(0)$

$3 = -3A$ so $A=-1$

when $x=3$: $3 = A(3-3) + B(3)$

$3 = 3B$ so $B=1$

$$\boxed{\frac{-1}{x} + \frac{1}{x-3}}$$

2. $\frac{x+1}{x^2-x-6} = \frac{x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$

multiplying by common denominator leaves just numerator on left and multiplies the right by "whats missing" from original denominator

$x+1 = A(x+2) + B(x-3)$

when $x=3$

$3+1 = A(3+2) + B(3-3)$

$4 = 5A$

$\frac{4}{5} = A$

when $x=-2$

$-2+1 = A(-2+2) + B(-2-3)$

$-1 = -5B$

$\frac{1}{5} = B$

$$\frac{4/5}{x-3} + \frac{1/5}{x+2} = \frac{4}{5(x-3)} + \frac{1}{5(x+2)}$$

3. $\frac{1}{4x^2-9} = \frac{1}{(2x-3)(2x+3)} = \frac{A}{2x-3} + \frac{B}{2x+3}$

$1 = A(2x+3) + B(2x-3)$

when $x = \frac{3}{2}$

$1 = A(2(\frac{3}{2})+3) + B(2(\frac{3}{2})-3)$

$1 = 6A$ so $A = \frac{1}{6}$

when $x = -3/2$

$1 = A(2(-\frac{3}{2})+3) + B(2(-\frac{3}{2})-3)$

$1 = -6B$ so $B = -\frac{1}{6}$

$$\frac{1}{6(2x-3)} - \frac{1}{6(2x+3)}$$

4. $\dfrac{6x^2-1}{x^2(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

LCD $= x^2(x-1)^2$

zeros $x=0, x=1$

$$6x^2-1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

when $x=0$

$$6(0)^2-1 = A(0)(0-1)^2 + B(0-1)^2 + C(0)^2(0-1) + D(0)^2$$

$$-1 = B$$

when $x=1$

$$6(1)^2-1 = A(1)(1-1)^2 + B(1-1)^2 + C(1)^2(1-1) + D(1)^2$$

$$5 = D$$

But how do we find A+C?

use $B=-1$, $D=5$ and random x values

use $x=-1$ (easy value)

$$6(-1)^2-1 = A(-1)(-1-1)^2 + -1(-1-1)^2 + C(-1)^2(-1-1) + 5(-1)^2$$

$$5 = -4A - 4 - 2C + 5 \Rightarrow 4 = -4A - 2C \quad *$$

try $x=2$ (easy value)

$$6(2)^2-1 = A(2)(2-1)^2 + -1(2-1)^2 + C(2)^2(2-1) + 5(2)^2$$

$$23 = 2A - 1 + 4C + 20 \Rightarrow 4 = 2A + 4C \quad *$$

$$\begin{cases} 4 = -4A - 2C \\ 4 = 2A + 4C \end{cases}$$

solve the system

$$8 = -8A - 4C$$
$$4 = 2A + 4C$$
$$12 = -6A$$
$$-2 = A$$

so $C = 2$

Answer:

$$\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1} + \frac{5}{(x-1)^2}$$

*This pick random x-values to use works but isn't necessarily the best way to approach the solution. The process of 'equating coefficients' is the most mathematically sound and can be used for any partial fraction decomposition.

5. $\dfrac{x^2-4x+7}{(x+1)\underbrace{(x^2-2x+3)}_{\text{irreducible}}} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-2x+3}$

$$x^2-4x+7 = A(x^2-2x+3)+(Bx+C)(x+1)$$

multiply fully

$$x^2-4x+7 = Ax^2-2Ax+3A+Bx^2+Bx+Cx+C$$

collect like terms

$$x^2-4x+7 = (A+B)x^2+(-2A+B+C)x+(3A+C)$$

equate coefficients of like powers of x

solve the system

$$\begin{cases} 1 = A+B & \rightarrow B = 1-A \\ -4 = -2A+B+C & \\ 7 = 3A+C & \rightarrow C = 7-3A \end{cases}$$

$$-4 = -2A+1-A+7-3A$$
$$-4 = -6A+8$$
$$-12 = -6A$$
$$2 = A \quad \text{so } B = 1-2 = -1 \text{ and } C = 7-3(2) = 7-6 = 1$$

Solution:

$$\frac{2}{x+1} + \frac{-x+1}{x^2-2x+3}$$