8.2 Operations with Matrices

Representation of Matrices –

1. A matrix can be denoted by any uppercase letter.

2. A matrix can be denoted by a representative element enclosed in brackets, such as $\left[a_{ij}\right]$.

3. A matrix can be denoted by a rectangular array of numbers.

Fact: Two matrices are equal if their corresponding entries are equal.

Examples: Find *x* and *y*.

1. $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$ $x = 13$ $y = 12$

2. $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

top row $5 = 2x+1$, $4 = 2x$, $2 = x$ middle row $6 = 3x$, $2 = x$ bottom row $4 = 3y-5$, $9 = 3y$, $3 = y$

This is good, x should be the same everywhere.

Definition of Matrix Addition – If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are matrices of order *m x n*, their sum is the *m x n* matrix given by $A + B = \left[a_{ij} + b_{ij}\right]$. The sum of two matrices of different orders is undefined.

Definition of Scalar Multiplication – If $A = \left[a_{ij}\right]$ is an *m x n* matrix and *c* is a scalar, the scalar multiple of A by *c* is the *m x n* matrix given by $cA = [ca_{ij}]$.

Properties of Matrix Addition and Scalar Multiplication – Let A, B, and C be *m x n* matrices and let *c* and *c* be scalars.

1. A + B = B + A — Commutative property of matrix addition
2. A + (B + C) = (A + B) + C — Associative property of matrix addition
3. (cd)A = c(dA) — Associative property of scalar multiplication
4. 1A = A — Scalar identity
5. c(A + B) = cA + cB — Distributive property
6. (c + d)A = cA + dA — Distributive property

Examples: If possible, find (a) A + B, (b) A – B, (c) 3A, and (d) 3A – 2B.

1. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

(a) $A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1+2 & -1+-1 \\ 2+-1 & -1+8 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1-2 & -1-(-1) \\ 2-(-1) & -1-8 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$

(c) $3A = 3\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(-1) \\ 3(2) & 3(-1) \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix}$

(d) $3A - 2B = 3\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -2 & 16 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 8 & -19 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+8 & 6+-1 \\ -1+2 & -5+3 \\ 1+-4 & 10+5 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 1 & -2 \\ -3 & 15 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1-8 & 6-(-1) \\ -1-2 & -5-3 \\ 1-(-4) & 10-5 \end{bmatrix} = \begin{bmatrix} -7 & 7 \\ -3 & -8 \\ 5 & 5 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 18 \\ -3 & -15 \\ 3 & 30 \end{bmatrix}$$

$$3A-2B = \begin{bmatrix} 3 & 18 \\ -3 & -15 \\ 3 & 30 \end{bmatrix} - \begin{bmatrix} 16 & -2 \\ 4 & 6 \\ -8 & 10 \end{bmatrix} = \begin{bmatrix} -13 & 20 \\ -7 & -21 \\ 11 & 20 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 7 & 2 \\ 9 & -1 \\ 5 & 0 \end{bmatrix}$

A+B not possible due to dimension difference

A−B not possible due to dimension difference

$3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$ 3A−2B not possible

Definition of Matrix Multiplication – If $A = \left[a_{ij}\right]$ is an *m x n* matrix and $B = \left[b_{ij}\right]$ is an *n x p* matrix, the product AB is an *m x p* matrix $AB = \left[c_{ij}\right]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj}$.

Basically, we multiply the row of A by the column of B, adding the values as we go.

Examples: If possible, find AB and state the order of the result.

1. $A=\begin{bmatrix}2 & 1\\ -3 & 4\\ 1 & 6\end{bmatrix}$, $B=\begin{bmatrix}0 & -1 & 0\\ 4 & 0 & 2\\ 8 & -1 & 7\end{bmatrix}$

$A = \underbrace{3\times 2 \quad 3}_{\text{not possible to multiply}}\times 3 = B$

not possible to multiply... but we can find BA.

$\overbrace{3\times 3 \quad 3\times 2}^{3\times 2 \text{ result}}$

Bonus: $BA = \begin{bmatrix} 0(2)-1(-3)+0(1) & 0(1)-1(4)+0(6) \\ 4(2)+0(-3)+2(1) & 4(1)+0(4)+2(6) \\ 8(2)-1(-3)+7(1) & 8(1)-1(4)+7(6)\end{bmatrix}$

$= \begin{bmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 46\end{bmatrix}$

2. $A=\begin{bmatrix}-1 & 6\\ -4 & 5\\ 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 0 & 9\end{bmatrix}$

$A \times B$

$3\times 2 \quad 2\times 2$ — ok — answer

$AB = \begin{bmatrix} -1(2)+6(0) & -1(3)+6(9) \\ -4(2)+5(0) & -4(3)+5(9) \\ 0(2)+3(0) & 0(3)+3(9)\end{bmatrix} = \begin{bmatrix} -2 & 51 \\ -8 & 33 \\ 0 & 27\end{bmatrix}$

3. $A=\begin{bmatrix}10\\ 12\end{bmatrix}$, $B=\begin{bmatrix}6 & -2 & 1 & 6\end{bmatrix}$

$A \times B$

$2\times \underbrace{1 \quad 1}_{ok}\times 4$ answer $= 2\times 4$

$AB = \begin{bmatrix} 10(6) & 10(-2) & 10(1) & 10(6) \\ 12(6) & 12(-2) & 12(1) & 12(6)\end{bmatrix} = \begin{bmatrix} 60 & -20 & 10 & 60 \\ 72 & -24 & 12 & 72\end{bmatrix}$

Examples: If possible, find (a) AB, (b) BA, and (c) A^2.

1. $A=\begin{bmatrix} 6 & 3 \\ -2 & -4 \end{bmatrix}$, $B=\begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 6(-2)+3(2) & 6(0)+3(4) \\ -2(-2)-4(2) & -2(0)-4(4) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ -4 & -16 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2(6)+0(-2) & -2(3)+0(-4) \\ 2(6)+4(-2) & 2(3)+4(-4) \end{bmatrix} = \begin{bmatrix} -12 & -6 \\ 4 & -10 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 6(6)+3(-2) & 6(3)+3(-4) \\ -2(6)-4(-2) & -2(3)-4(-4) \end{bmatrix} = \begin{bmatrix} 30 & 6 \\ -4 & 10 \end{bmatrix}$$

2. $A=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1(1)-1(-3) & 1(3)-1(1) \\ 1(1)+1(-3) & 1(3)+1(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1(1)+3(1) & 1(-1)+3(1) \\ -3(1)+1(1) & -3(-1)+1(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1(1)-1(1) & 1(-1)-1(1) \\ 1(1)+1(1) & 1(-1)+1(1) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Notice that in example 2, $AB = BA$ but in example 1 that does not hold true. This leads us to some properties of matrix multiplication...

Properties of Matrix Multiplication –Let A, B, and C be matrices and let *c* be a scalar.

1. A(BC) = (AB)C	Associative property of matrix multiplication.
2. A(B + C) = AB + AC	Distributive property
3. (A + B)C = AC + BC	Distributive property
4. c(AB) = (cA)B = A(cB)	Associative property of scalar multiplication

There is <u>not</u> a commutative property stating $AB = BA$

Definition of Identity Matrix – The *n x n* matrix that consists of 1's on the diagonal and 0's elsewhere is called the identity matrix of order *n x n*. Not that the identity matrix must be square.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{and so on}$$