8.3 The Inverse of a Square Matrix

Definition of the Inverse of a Square Matrix – Let A be an *n x n* matrix and let I_n be the *n x n* identity matrix. If there exists a matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$, then A^{-1} is called the inverse of A.

Examples: Show that B is the inverse of A.

1. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ we must show that $AB = I_2 = BA$

$$AB = \begin{bmatrix} 1(2)-1(1) & 1(1)-1(1) \\ -1(2)+2(1) & -1(1)+2(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

this is not enough, it has to be shown both ways. ugh!

$$BA = \begin{bmatrix} 2(1)+1(-1) & 2(-1)+1(2) \\ 1(1)+1(-1) & 1(-1)+1(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

they are inverses!

2. $A = \begin{bmatrix} -4 & 1 & 5 \\ -1 & 2 & 4 \\ 0 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{4} & -1 & -\frac{11}{4} \\ -\frac{1}{4} & 1 & \frac{7}{4} \end{bmatrix}$

$$AB = \begin{bmatrix} -4(-\frac{1}{2})+1(\frac{1}{4})+5(-\frac{1}{4}) & -4(1)+1(-1)+5(1) & -4(\frac{3}{2})+1(-\frac{11}{4})+5(\frac{7}{4}) \\ -1(-\frac{1}{2})+2(\frac{1}{4})+4(-\frac{1}{4}) & -1(1)+2(-1)+4(1) & -1(\frac{3}{2})+2(-\frac{11}{4})+4(\frac{7}{4}) \\ 0(-\frac{1}{2})-1(\frac{1}{4})-1(-\frac{1}{4}) & 0(1)-1(-1)-1(1) & 0(\frac{3}{2})-1(-\frac{11}{4})-1(\frac{7}{4}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -\frac{1}{2}(-4)+1(-1)+\frac{3}{2}(0) & -\frac{1}{2}(1)+1(2)+\frac{3}{2}(1) & -\frac{1}{2}(5)+1(4)+\frac{3}{2}(-1) \\ \frac{1}{4}(-4)-1(-1)-\frac{11}{4}(0) & \frac{1}{4}(1)-1(2)-\frac{11}{4}(-1) & \frac{1}{4}(5)-1(4)-\frac{11}{4}(-1) \\ -\frac{1}{4}(-4)+1(-1)+\frac{7}{4}(0) & -\frac{1}{4}(1)+1(2)+\frac{7}{4}(-1) & -\frac{1}{4}(5)+1(4)+\frac{7}{4}(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding an Inverse Matrix – Let A be a square matrix of order *n*.

1. Write the *n x 2n* matrix that consists of the given matrix A on the left and the identity matrix augmented on the right.

2. If possible, row reduce A to I using elementary row operations on the entire matrix $\left[A \mid I\right]$. The result will be the matrix $\left[I \mid A^{-1}\right]$. If this is not possible, A is not invertible.

3. Check your work by multiplying to see that the definition holds.

Examples: Find the inverse of the matrix, if it exists.

1. $A=\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$

$$[A \mid I_2] = \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 3 & 7 & 0 & 1\end{array}\right] \qquad \text{Goal: } \left[\begin{array}{cc|cc}1 & 0 & \# & \#\\ 0 & 1 & \# & \#\end{array}\right] \leftarrow A^{-1}$$

$-3R1$	-3	-6	-3	0
$+R2$	3	7	0	1
New R2	0	1	-3	1

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & 1 & -3 & 1\end{array}\right]$$

$-2R2$	0	-2	6	-2
$+R1$	1	2	1	0
New R1	1	0	7	-2

$$\left[\begin{array}{cc|cc}1 & 0 & 7 & -2\\ 0 & 1 & -3 & 1\end{array}\right] \qquad A^{-1}=\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}$$

multiply to verify

2. $A=\begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}$

$$[A \mid I_2] = \left[\begin{array}{cc|cc}4 & -1 & 1 & 0\\ -3 & 1 & 0 & 1\end{array}\right]$$

$\frac{1}{4}R_1 = \text{new } R1$

$$\left[\begin{array}{cc|cc}1 & -1/4 & 1/4 & 0\\ -3 & 1 & 0 & 1\end{array}\right]$$

$3R1$	3	$-3/4$	$3/4$	0
$+R2$	-3	1	0	1
new R2	0	$1/4$	$3/4$	1

$$\left[\begin{array}{cc|cc}1 & -1/4 & \frac{1}{4} & 0\\ 0 & 1/4 & 3/4 & 1\end{array}\right] \quad 4R_2 = \text{new } R2 \quad \left[\begin{array}{cc|cc}1 & -1/4 & \frac{1}{4} & 0\\ 0 & 1 & 3 & 4\end{array}\right]$$

$\frac{1}{4}R2$	0	$\frac{1}{4}$	$\frac{3}{4}$	1
$+R1$	1	$-\frac{1}{4}$	$\frac{1}{4}$	0
new R1	1	0	1	1

$$\left[\begin{array}{cc|cc}1 & 0 & 1 & 1\\ 0 & 1 & 3 & 4\end{array}\right] \qquad A^{-1}=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$$

Check by multiplying

There has to be a better/easier way.

Fact: To find the inverse of a 2 x 2 matrix we can use a special formula. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Examples: Find the inverse using the formula.

1. $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

$ad-bc = 1(2) - (-2)(-3) = 2 - 6 = -4$

$A^{-1} = \frac{1}{-4}\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -3/4 & -1/4 \end{bmatrix}$

multiply to verify

switch places

Change sign

2. $A = \begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

$ad-bc = -12(-2) - 3(5) = 24 - 15 = 9$

$A^{-1} = \frac{1}{9}\begin{bmatrix} -2 & -3 \\ -5 & -12 \end{bmatrix} = \begin{bmatrix} -2/9 & -1/3 \\ -5/9 & -4/3 \end{bmatrix}$

3. You try it: $A = \begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$