# Chapter 10 Introduction to the Derivative

The concept of a derivative takes up half the study of Calculus. A derivative, basically, represents rates of change.

## *10.1 Limits: Numerical and Graphical Approaches*

Rates of change are calculated by derivatives, but an important part of the definition of the derivative is something called a limit.

Ex: Take the function $f(x) = -2x+1$ and ask what happens to f(x) as x approaches 2?

| x | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| f(x) | -2.8 | -2.98 | -2.998 | ? | -3.002 | -3.02 | -3.2 |

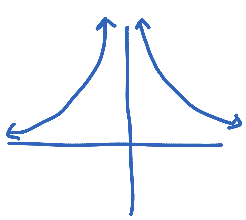
approaches -3      approaches -3

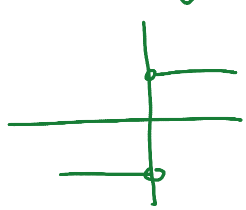
We can conclude that at x=2, the function is near -3

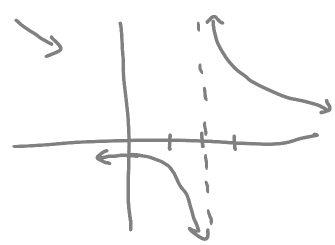
* We don't plug in the value we are asking about because the question asks what happens as we approach that value, it doesn't ask what's happening AT that value. We could do this for any function. *

**Definition:** If f(x) approaches the number L as x approaches (but is not equal to) *a* from both sides, then we say that f(x) approaches L as x→a or that the limit of f(x) as x→a is L. We write $\lim_{x\to a} f(x) = L$ or f(x)→L as x→a. If f(x) fails to approach a single fixed number as x approaches *a* from both sides, then we say that f(x) has no limit as x→a, or $\lim_{x\to a} f(x)$ does not exist.

Ex: Limits that fail include $\lim_{x\to 0} \frac{1}{x^2}, \quad \lim_{x\to 0} \frac{|x|}{x}, \quad and \quad \lim_{x\to 2} \frac{1}{x-2}$







In another useful kind of limit, we let x approach either ±∞, by which we mean that we let x get arbitrarily large in either direction.

Ex: Find $\lim_{x\to\infty} \dfrac{3x^2-2}{x^2+7}$ A table is useful.

| x | 1 | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|---|
| f(x) | 0.125 | 2.785 | 2.998 | 3 | 3 |

(2.999977..)

So $\lim_{x\to\infty} f(x) = 3$

The graph allows us to see limits without having to evaluate a bunch of numbers.

**Summary of Graphical Method:** To decide whether $\lim_{x\to a} f(x)$ exists and to find its value if it does:

1. Draw the graph of f(x) by hand or with graphing technology.

2. Position your pencil point (or the Trace cursor) on a point of the graph to right of x=a.

3. Move the point along the graph toward x=a from the right and read the y-coordinate as you go. The value the y-coordinate approaches (if any) is the limit $\lim_{x\to a^+} f(x)$. (The limit from the right.)

4. Repeat steps 2 and 3, this time starting from a point on the graph to the left of x=a, and approaching x=a along the graph from the left. The value the y-coordinate approaches (if any) is $\lim_{x\to a^-} f(x)$. (The limit from the left.)

5. If the left and right limits both exist and have the same value L, then $\lim_{x\to a} f(x) = L$ Otherwise, the limit does not exist. The value f(a) has no relevance whatsoever.

6. To evaluate $\lim_{x\to\infty} f(x)$, move the pencil point toward the far right of the graph and estimate the value the y-coordinate approaches (if any). For $\lim_{x\to-\infty} f(x)$, move the pencil point toward the far left.

7. If x=a happens to be an endpoint of the domain of $f(x)$, then only a one-sided limit is possible at x=a. For instance, if the domain is (-∞,4], then $\lim_{x\to 4^-} f(x)$ can be computed, but not the limit from the right and therefore not the limit itself.

## *10.2 Limits and Continuity*

As we saw in a couple of graphs from section 10.1, graphs could have different behaviors on either side of the value we are trying to learn about with the limit. For this reason we sometimes consider one-sided limits. In order to discuss one-sided limits we need special notation.

**My Definition:** On a number line, negative values are to the left. For this reason we write the limit as *x* approaches *a* from the left of *f(x)* as $\lim_{x\to a^-} f(x)$. The raised negative sign after the *a* indicates that we are only considering values to the left of *a.* Similarly we can define $\lim_{x\to a^+} f(x)$ as the limit as *x* approaches *a* from the right only. Tip: I may also refer to the left as "below" and right as "above" once again in reference to the number line.

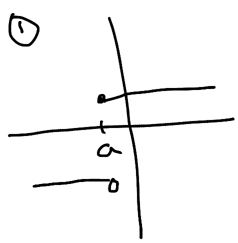
**My Fact:** When we say a limit exists, we are saying that the limit from the right and the limit from the left are equal.

**Definition.** Let *f* be a function and let *a* be a number in the domain of *f*. Then *f* is continuous at *a* if

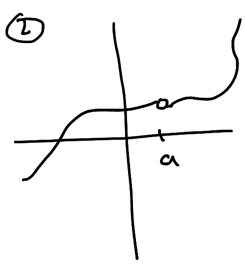
a) $\lim_{x\to a} f(x)$ exists, and b) $\lim_{x\to a} f(x) = f(a)$

The function *f* is said to be continuous on its domain if it is continuous at each point in its domain. If *f* is not continuous at a particular *a* in its domain, we say that *f* is discontinuous at *a* or that *f* has a discontinuity at *a*. Thus, a discontinuity can occur at *x=a* if either the limit does not exist or if the limit exists but is not equal to *f(a).*

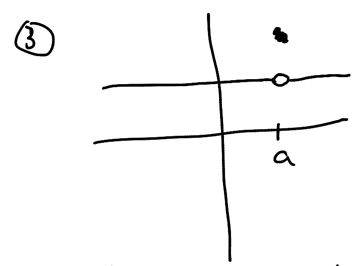
Example: How continuity can fail.




limit does not exist



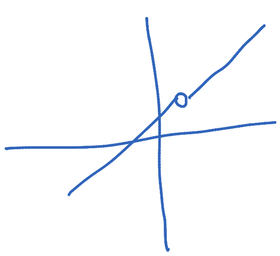

f(a) does not exist



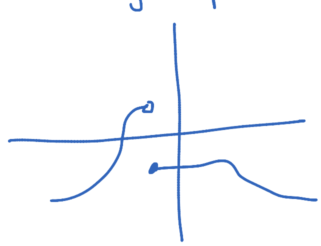

limit exists and f(a) exists but they are not equal.

Example: Types of discontinuities.

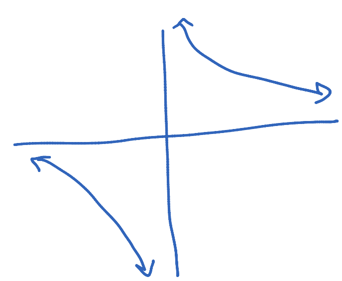
removable



jump



infinite



**Fact**: Some functions are not defined at *x=a* but through the use of limits we can find a suitable definition in order to make them continuous. These functions are said to have a removable singularity at *a*. If we are unable to "fix" the discontinuity, the function has an essential singularity there.

Examples. Find an appropriate value, if one exists, to make *f* continuous at *x=a*.

1. $f(x)=\dfrac{x^2+3x+2}{x+1}; a=-1$ When we substitute 1 for x we get $\frac{0}{0}$. This tells us we must use algebra.

$$f(x)=\frac{x^2+3x+2}{x+1}=\frac{(x+1)(x+2)}{x+1}=x+2$$

If we look at the simplified function at $-1$ we get that f would be $-1+2=1$. Therefore we define $f(-1)=1$ so that f is continuous.

With this definition we are filling the hole in the graph.

2. $f(x)=\dfrac{x^2-3x}{x+4}; a=-4$ $\quad f(-4)=\dfrac{(-4)^2-3(-4)}{-4+4}=\dfrac{16+12}{0}=\dfrac{28}{0}$

There is no magical algebra trick that can help when only the denominator is zero. This tells us there is no way to make f continuous.

3. $f(x)=\dfrac{x-1}{x^3-1}; a=1$ Always evaluate first: $f(1)=\frac{0}{0}\rightarrow$ Something can be done

Factor: $f(x)=\dfrac{x-1}{(x-1)(x^2+x+1)}=\dfrac{1}{x^2+x+1}$

Evaluate the new function: $f(1)=\dfrac{1}{(1)^2+(1)+1}=\dfrac{1}{3}$.

So if $f(1)=\frac{1}{3}$, $f(x)=\dfrac{x-1}{x^3-1}$ will be continuous.

## *10.3 Limits and Continuity: Algebraic Approach*

Making a chart or drawing a graph is ok. However, both methods are time consuming and are better off with the help of technology. There must be a better way; and there is, it's called algebra!

**Definition:** A closed-form function is any function that can be obtained by combining constants, powers of x, exponential functions, radicals, logarithms, and trigonometric functions (and some other functions we do not encounter in this text) into a single mathematical formula by means of the usual arithmetic operations and composition of functions.

Basically, everything except piece-wise defined functions are closed-form, as far as this course is concerned.

**Theorem** Every closed form function is continuous on its domain. Thus, if *f* is a closed form function and *f(a)* is defined, we have $\lim_{x\to a} f(x) = f(a)$.

**Fact** If *f(x) = g(x)* for all *x* except possibly *x = a*, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Note: We used this fact in the last section when we factored and canceled to try and fill in the discontinuity.

**Theorem** If *f(x)* has the form $f(x) = \dfrac{c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0}{d_m x^m + d_{m-1}x^{m-1} + \ldots + d_1 x + d_0}$ with the $c_i$ and $d_i$ constants ( $c_n \neq 0$ and $d_m \neq 0$ ), then we can calculate the limit of *f(x)* as *x*→±∞ by ignoring all powers of *x* except the highest in both the numerator and denominator. Thus, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{c_n x^n}{d_m x^m}$.

## Strategy

If *a* is a finite number:

1. Decide whether *f* is a closed form function. If it is not, then find the left and right limits at the values of *x* where the function changes from one formula to another.

2. If f is a closed form function, try substituting *x*=*a* in the formula for f(x). Then one of three things will happen:

a) f(a) is defined. Then $\lim_{x\to a} f(x) = f(a)$.

b) *f*(*a*) is not defined and has the form 0/0. Try to simplify the expression *f* to cancel one of the terms that gives 0.

c) f(a) is not defined and has the form k/0 where *k* is not zero. Then the function diverges to ±∞ as *x* approaches *a* from each side. Check this graphically.

If *a*=±∞: If the given function is a polynomial or ratio of polynomials, use the theorem and focus only on the highest powers of *x*.

Examples: Calculate the limit. If a limit does not exist, state why.

1. $\lim_{x\to 0}(2x-4) = 2(0)-4 = \boxed{-4}$

a=0 is finite, f(x)=2x-4 is a closed form function

2. $\lim_{x\to -1}\dfrac{4x^2+1}{x} = \dfrac{4(-1)^2+1}{(-1)} = \dfrac{4+1}{-1} = \boxed{-5}$

3. $\lim_{h\to 0}\dfrac{h^2+h}{h^2+2h} = \lim_{h\to 0}\dfrac{h+1}{h+2} = \dfrac{0+1}{0+2} = \boxed{\dfrac{1}{2}}$

evaluation would give $\frac{0}{0}$ so first simplify

$\dfrac{h^2+h}{h^2+2h} = \dfrac{\cancel{h}(h+1)}{\cancel{h}(h+2)} = \dfrac{h+1}{h+2}$

4. $\lim_{x\to 0^+} \frac{1}{x^2 - x} = \frac{1}{0}$ ? Not an acceptable answer.

right

| $x$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|
| $f(x)$ | ? | $-1001$ | $-101$ | $-11$ |

$\lim_{x\to 0^+} \frac{1}{x^2 - x} = -\infty$

5. $\lim_{x\to\infty} \frac{6x^2 + 5x + 100}{3x^2 - 9} = \lim_{x\to\infty} \frac{6x^2}{3x^2} = \lim_{x\to\infty} \frac{6}{3} = \frac{6}{3} = \boxed{2}$

According to the theorem we can just focus on the leading terms.

And now we can simplify

There is no x left to substitute so the limit is the constant, simplified

6. $\lim_{x\to -\infty} \frac{2x^4 + 20x^3}{1000x^3 + 6} = \lim_{x\to -\infty} \frac{2x^4}{1000x^3} = \lim_{x\to -\infty} \frac{x}{500} = \frac{-\infty}{500}$ which is not the final answer but it leads to $-\frac{\infty}{500} = \boxed{-\infty.}$

Examples: Find all points of discontinuity.

1. $f(x) = \begin{cases} 1-x, & if\ x \le 1 \\ x-1, & if\ x > 1 \end{cases}$

Both pieces are closed form so each piece is good. The only possible discontinuity is at the cut point of $x=1$.

Left $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$

Right $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 1-1 = 0$

Since the limit from the left and the limit from the right are equal, the limit exists. The limit equals $f(1)$ also so is continuous everywhere.

2. $g(x)=\begin{cases} x^3+2, & \text{if } x\le -1 \\ x^2, & \text{if } -1<x<0 \\ x, & \text{if } x\ge 0 \end{cases}$

Once again we focus on the cut points.

For $x=-1$: left $\lim_{x\to -1^-} g(x)=\lim_{x\to -1^-}(x^3+2)=(-1)^3+2=-1+2=1$

right $\lim_{x\to -1^+} g(x)=\lim_{x\to -1^+}(x^2)=(-1)^2=1$

$\Bigg\}$ continuous at $x=-1$

For $x=0$. left $\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-} x^2=(0)^2=0$

right $\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+} x=0$

$\Bigg\}$ continuous at $x=0$

This function is continuous everywhere.

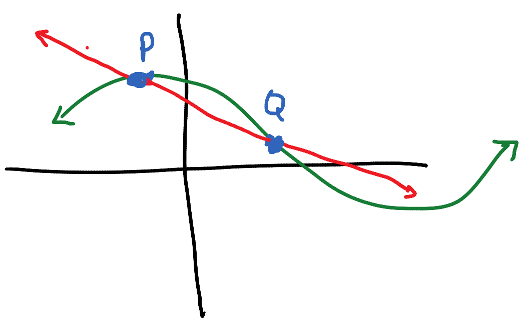
Note: If you find a function has different limits from the left and right it will be discontinuous at that cut point.

## *10.4 Average Rate of Change*

**Definition:** The change in *f*(*x*) over the interval [*a,b*] is: Change in *f* = *Δf* = Second value – First value = *f(b) - f(a*)

The average rate of change of *f(x)* over the interval [a,b] is $\frac{Change\,in\,f}{Change\,in\,x} = \frac{\Delta f}{\Delta x} = \frac{f(b)-f(a)}{b-a}$ = Slope of line through points $P(a, f(a))$ and $Q(b, f(b))$.

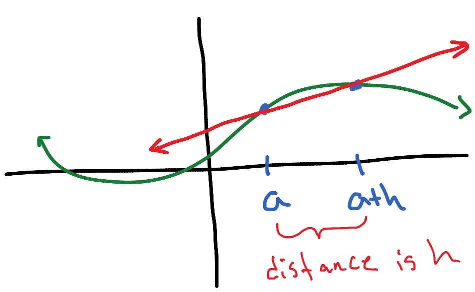
We also call this average rate of change the difference quotient of *f* over the interval *[a,b*]. A line through two points of a graph like P and Q is called a secant line of the graph.




The secant line shows the average rate of change (slope) between points P and Q.

**Fact** The units of the change in *f* are the units of f(x). The units of the average rate of change of *f* are units of *f*(*x*) per unit of *x*.
divided by

**Alternate Formula** The average rate of change of *f* over [*a,a+h*] is $\frac{f(a+h)-f(a)}{h}$.




We have many ways of representing data and therefore functions. We may use a table of values, graphs, or algebra as condensed ways of showing a function. All these ways will allow us to find average rates of change.

Examples: Calculate the average rate of change of the given function over the given interval. Include correct units.

1. Find the average rate of change over the interval [0,2].

| *x* | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| *f(x)* | -1 | 3 | 2 | 1 |

avg r.o.c. is $\frac{f(2)-f(0)}{2-0} = \frac{2-(-1)}{2-0} = \frac{2+1}{2} = \frac{3}{2}$

2. Find the average rate of change over the interval [0.1,0.2].

| *t (hours)* | 0 | 0.1 | 0.2 |
|---|---|---|---|
| *D(t) (miles)* | 0 | 3 | 6 |

avg r.o.c. is $\frac{D(0.2)-D(0.1)}{0.2-0.1} = \frac{6-3}{0.2-0.1} = \frac{3}{0.1} = 30$ mph

3. $f(x) = 2x^2 + 4, [-1,2]$

$f(2) = 2(2)^2 + 4$
$= 2(4) + 4 = 12$

$f(-1) = 2(-1)^2 + 4$
$= 2(1) + 4 = 6$

avg r.o.c. is $\frac{f(2)-f(-1)}{2-(-1)} = \frac{12-6}{2+1} = \frac{6}{3} = 2$

4. $f(x) = 3x^2 + \frac{x}{2}, \; [3,4]$

$f(4) = 3(4)^2 + \frac{4}{2}$
$= 3(16) + 2 = 50$

$f(3) = 3(3)^2 + \frac{3}{2}$
$= 3(9) + \frac{3}{2} = 28.5$

avg r.o.c. is $\frac{f(4)-f(3)}{4-3} = \frac{50-28.5}{1}$

$= 21.5$

Examples: Calculate the average rate of change of $f(x) = \frac{2}{x}$ over the intervals [1,1+h] where h=1,h=0.1,h=0.01,h=0.001.

First we use the formula in general $\frac{f(a+h)-f(a)}{h} = \frac{f(1+h)-f(1)}{h}$

$$= \frac{\frac{2}{1+h} - \frac{2}{1}}{h} = \frac{\frac{2}{1+h} - \frac{2}{1}\cdot\frac{1+h}{1+h}}{h} = \frac{\frac{2-2(1+h)}{1+h}}{h} = \frac{-2h}{1+h}\cdot\frac{1}{h} = -\frac{2}{1+h}$$

dividing by h is multiplying by $\frac{1}{h}$

Now we evaluate

| $h$ | 1 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|
| $-\frac{2}{1+h}$ | $-1$ | $-1.818$ | $-1.980$ | $-1.998$ |

Example: The following table lists the net sales (after-tax revenue) at the Finnish cell phone company Nokia during the period 1997-2003 (t=0 represents 2000):

| Year, *t* | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| Nokia sales *P(t)* in € Billion | 9 | 14 | 20 | 31 | 31 | 30 | 29 |

Compute **and interpret** the average rate of change of P(t)

a. over the period [-2,3], and

$$\frac{P(3)-P(-2)}{3-(-2)} = \frac{29-14}{5} = \frac{15}{5} = 3 \text{ € Billion/year}$$

Nokia gained an avg of 3 billion Euros per ~~day~~ year from 1998 to 2003

b. over the period [0,1].

$$\frac{P(1)-P(0)}{1-0} = \frac{31-31}{1} = \frac{0}{1} = 0 \text{ € Billion/year}$$

Nokia did not make any gains or losses from 2000 to 2001. (Sales remained constant)