10.5 Derivatives: Numerical and Graphical Viewpoints

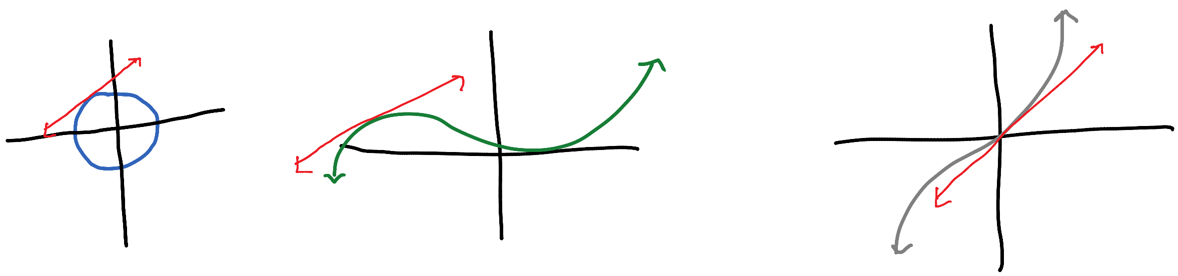
Definition: The instantaneous rate of change of f(x) at *x* = *a* is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The quantity *f′(a)* is also called the derivative of *f(x)* at *x* = *a*. Finding the derivative is also known as differentiating *f*. The units of *f′(a)* are the same as the units of the average rate of change: units of *f* per unit of *x*.

If this limit does not exist, for whatever reason, we say that *f* is not differentiable at *x* = *a*, or *f′(a)* does not exist.

A tangent line to a circle is a line that touches the circle in just one point. A tangent line gives the circle "a glancing blow." For a smooth curve other than a circle, a tangent line may touch the curve at more than one point, or pass through it. However, all tangent lines have the following interesting property in common: If we focus on a small portion of the curve very close to the point P the curve will appear almost straight, and almost indistinguishable from the tangent line.



Secants and Tangents

The slope of the secant line through the points on the graph of *f* where *x* = *a* and *x* = *a*+*h* is given by the average rate of change, or difference quotient, $m_{\sec}$ = slope of secant = average rate of change = $\frac{f(a+h)-f(a)}{h}$.

The slope of the tangent line through the point on the graph of *f* where *x* = *a* is given by the instantaneous rate of change, or derivative $m_{\tan}$ = slope of tangent = derivative =

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Notice that this is saying that the slope of the tangent line is a good way to estimate the derivative at a certain point graphically.

Example: Estimate the derivative of *r(x)* from the table of average rates of change. *r'(5)* = ________

h	1	0.1	0.01	0.001	0.0001
Avg r.o.c. of r over [5,5+h]	4.5	6.44	6.474	6.49928	6.4999990
h	-1	-0.1	-0.01	-0.001	-0.0001
Avg r.o.c. of r over [5+h,5]	8.0	6.54	6.528	6.50044	6.5000066

As h gets really small and we get closer and closer to 5 we see the avg rate of change gets close to 6.5. This means $r'(5) = 6.5$

Example: Consider the function $R(t) = 60t - 3t^2$ as representing the value of an ounce of palladium in U.S. dollars as a function of the time *t* in days. Find the average rate of change of *R(t)* over the time intervals [3, 3+h], where *h = 1, 0.1, and 0.01* days. Round your answer to two decimal places.

h	1	0.1	0.01
Avg r.o.c.	39.00	41.70	41.97

Next, estimate the instantaneous rate of change of *R* at time *t*, specifying the units of measurement.

$$\frac{R(t+h)-R(t)}{h} = \frac{60(t+h)-3(t+h)^2 - (60t-3t^2)}{h} = \frac{60t+60h-3(t^2+2th+h^2)-60t+3t^2}{h}$$

$$= \frac{\cancel{60t}+60h-\cancel{3t^2}-6th-3h^2-\cancel{60t}+\cancel{3t^2}}{h} = \frac{60h-6th-3h^2}{h} = 60-6t-3h$$

With $t=3$ we have avg r.o.c is $60-6(3)-3h = 42-3h$

$h=1$: $42-3(1) = 39$

$h=0.1$: $42-3(0.1) = 41.7$

$h=0.01$: $42-3(0.01) = 41.97$

As h gets smaller, it looks like avg r.o.c gets closer to 42, the instantaneous r.o.c

That is, $R'(3) = \$42/\text{day}$.

Approximating We can calculate an approximate value of *f′(a)* by using the formula $f'(a) \approx \frac{f(a+h)-f(a)}{h}$ with a small value of *h*. The value h = 0.0001 often works. The following alternative formula, which measures the rate of change of *f* over the interval [*a-h,a+h*], often gives a more accurate result, and is the one used in many calculators: $f'(a) = \frac{f(a+h)-f(a-h)}{2h}$.

Examples: Estimate each derivative in two ways. Round your answer to three decimal places.

1. $f(x) = \frac{x}{5} - 6$ when *x = -4*. Find *f′(-4)*

① $f'(-4) \approx \frac{f(-4+0.0001)-f(-4)}{0.0001} = \frac{-6.79998-(-6.8)}{0.0001} = 0.2$

② $f'(-4) = \frac{f(-4+0.0001)-f(-4-0.0001)}{2(0.0001)} = \frac{-6.79998-(-6.80002)}{0.0002} = 0.2$

2. $y = 7 - x^2$; estimate $\left.\frac{dy}{dx}\right|_{x=-5}$

$\left.\frac{dy}{dx}\right|_{x=-5} = y'(-5) \approx \frac{y(-5+0.0001)-y(-5)}{0.0001} = \frac{-17.999-(-18)}{0.0001} \approx 9.9999$

$= 10.000$ rounded

$\left.\frac{dy}{dx}\right|_{x=-5} = \frac{y(-5+0.0001)-y(-5-0.0001)}{2(0.0001)} = \frac{-17.999-(-18.001)}{0,0002} = 10$

Notation Two guys came up with calculus independently about 16 years apart. These guys were Newton and Leibniz. Today we use parts of each theory. Recall that the average rate of change was $\frac{\Delta f}{\Delta x}$ and instantaneous rate of change would then be $\lim_{h\to 0}\frac{\Delta f}{\Delta x}$, but delta is a Greek letter and Leibniz was German so he used $\frac{df}{dx} = \lim_{h\to 0}\frac{\Delta f}{\Delta x}$. NOTE: $\frac{df}{dx}$ is not a fraction, it is a whole unit unto itself that does not follow the rules of fractions and such. The $\frac{d}{dx}$ portion is called an operator, it is telling you to find the derivative of *f* with *x* as your variable.

The Derivative Function

The derivative *f′(x)* is a number we can calculate, or at least approximate, for various values of *x*. Because *f′*(*x*) depends on the value of *x*, we may think of *f′* as a function. This function is the derivative function.

Definition: If *f* is a function, its derivative function *f′(x)* is given by $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Definition. For an object moving in a straight line with position *s(t)* at time *t*, the average velocity from time *t* to time *t+h* is the average rate of change of position with respect to time:

$$v_{avg}=\frac{s(t+h)-s(t)}{h}=\frac{\Delta s}{\Delta t}.$$

The instantaneous velocity at time *t* is $v=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}=\frac{ds}{dt}$. In other words, instantaneous velocity is the derivative of position with respect to time.

Example: If a stone is thrown down at 80 ft/sec from a height of 950 feet, its height after *t* seconds is given by $s=950-80t-16t^2$.

a) Find its average velocity over the period [1, 5].

$$V_{avg}=\frac{s(5)-s(1)}{5-1}=\frac{150-854}{4}=\frac{-704}{4}=-176\text{ ft/sec}$$

b) Estimate its instantaneous velocity at time *t* = 5.

$$V_{inst}=\lim_{h\to 0}\frac{s(5+h)-s(5)}{h}=\lim_{h\to 0}\frac{950-80(5+h)-16(5+h)^2-150}{h}$$

$$=\lim_{h\to 0}\frac{950-400-80h-16(25+10h+h^2)-150}{h}$$

$$=\lim_{h\to 0}\frac{950-400-80h-400-160h-16h^2-150}{h}$$

$$=\lim_{h\to 0}\frac{-16h^2-240h}{h}=\lim_{h\to 0}(-16h-240)=-16(0)-240=-240\text{ ft/sec}$$

10.6 The Derivative: Algebraic Viewpoint

The definition of the derivative from the last section allows us to compute the derivative algebraically without having to estimate with tables or graphs. How? Easy!

Example: Let $f(x) = 3x^2 - 4$. Find $f'(2)$. Why 2 specifically? It's not magic. Find $f'(x)$ in general.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h)^2 - 4 - (3(2)^2 - 4)}{h} = \lim_{h\to 0} \frac{3(4+4h+h^2) - 4 - 8}{h}$$

$$= \lim_{h\to 0} \frac{12+12h+3h^2-4-8}{h} = \lim_{h\to 0} \frac{+12h+3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(+12+3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (+12+3h) = +12 + 3(0) = +12. \text{ That is, } f'(2) = +12$$

In general we must find $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. Let's do some preliminary work first.

(1)

$$\begin{aligned} f(x+h) &= 3(x+h)^2 - 4 \\ &= 3(x^2+2xh+h^2) - 4 \\ &= 3x^2 + 6xh + 3h^2 - 4 \end{aligned}$$

Next:

(2)

$$\begin{aligned} f(x+h) - f(x) &\\ &= 3x^2+6xh+3h^2-4-(3x^2-4) \\ &= 3x^2+6xh+3h^2-4-3x^2+4 \\ &= 6xh+3h^2 \end{aligned}$$

Now the limit is easier:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}} = \lim_{h\to 0} (6x+3h)$$

$$= 6x + 3(0) = 6x. \text{ That is, } f'(x) = 6x.$$

Notice that with this formula we still get $f'(2) = 6(2) = 12$.

Examples: Compute *f′(a)* algebraically.

1. $f(x) = -2x + 4; a = -1$

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{6-2h-6}{h} = \lim_{h\to 0} \frac{-2h}{h} = \lim_{h\to 0} -2 = -2$$

$$f(-1+h) = -2(-1+h) + 4 = +2 - 2h + 4 = 6 - 2h$$

$$f(-1) = -2(-1) + 4 = 2 + 4 = 6$$

so $\boxed{f'(-1) = -2}$

2. $f(x) = 2x^2 + x; a = -2$

$$f(-2+h) = 2(-2+h)^2 + (-2+h) = 2(4 - 4h + h^2) - 2 + h = 8 - 8h + 2h^2 - 2 + h$$
$$= 6 - 7h + 2h^2$$

$$f(-2) = 2(-2)^2 + (-2) = 2(4) - 2 = 6$$

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{6 - 7h + 2h^2 - 6}{h} = \lim_{h\to 0} \frac{-7h + 2h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-7+2h)}{\cancel{h}} = \lim_{h\to 0} (-7+2h) = -7 + 2(0) = \boxed{-7}$$

Examples: Compute *f′(x)* algebraically.

1. $f(x) = x - x^2$

$$f(x+h) = (x+h) - (x+h)^2$$
$$= x + h - (x^2 + 2xh + h^2)$$
$$= x + h - x^2 - 2xh - h^2$$

$$f(x+h) - f(x) = x + h - x^2 - 2xh - h^2 - (x - x^2)$$
$$= \underline{x} + h - \underline{\underline{x^2}} - 2xh - h^2 \underline{-x} + \underline{\underline{x^2}}$$
$$= h - 2xh - h^2$$
$$= h(1 - 2x - h)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(1-2x-h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (1 - 2x - h) = 1 - 2x - (0)$$
$$= 1 - 2x$$

so $\boxed{f'(x) = 1 - 2x}$

2. $f(x) = x - 2x^3$

$f(x+h) = (x+h) - 2(x+h)^3$
$= x + h - 2(x^3 + 3x^2h + 3xh^2 + h^3)$
$= x + h - 2x^3 - 6x^2h - 6xh^2 - 2h^3$

$f(x+h) - f(x) = h - 6x^2h - 6xh^2 - 2h^3$
$= h(1 - 6x^2 - 6xh - 2h^2)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h\to 0} \frac{h(1-6x^2-6xh-2h^2)}{h}$
$= \lim_{h\to 0} (1-6x^2-6xh-2h^2)$
$= 1-6x^2-6x(0)-2(0)^2$
$= \boxed{1-6x^2}$

3. $f(x) = \frac{2}{x}$

$f(x+h) = \frac{2}{x+h}$

$f(x+h) - f(x) = \frac{2}{x+h} - \frac{2}{x}$
$= \frac{2}{x+h} \cdot \frac{x}{x} - \frac{2}{x} \cdot \frac{x+h}{x+h}$
$= \frac{2x}{x(x+h)} - \frac{2(x+h)}{x(x+h)}$
$= \frac{2x-2x-2h}{x(x+h)} = \frac{-2h}{x(x+h)}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h\to 0} \frac{\frac{-2h}{x(x+h)}}{h}$

divide by h is the same as multiply by $\frac{1}{h}$

$= \lim_{h\to 0} \frac{-2h}{x(x+h)} \cdot \frac{1}{h}$
$= \lim_{h\to 0} \frac{-2}{x(x+h)} = \frac{-2}{x(x+0)} = \boxed{\frac{-2}{x^2}}$

Examples: Find the equation of the tangent to the graph at the indicated point.

$y - f(a) = f'(a)(x-a)$

1. $f(x) = x^2 + 1; a = 2$ *

$f(2+h) = (2+h)^2 + 1 = 4 + 4h + h^2 + 1$
$= 5 + 4h + h^2$

$f(2) = (2)^2 + 1 = 4 + 1 = 5$ *

$f'(2) = \lim_{h\to 0} \frac{\cancel{5} + 4h + h^2 - \cancel{5}}{h} = \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}} = \lim_{h\to 0} (4+h) = 4$ *

$y - 5 = 4(x-2)$
$y - 5 = 4x - 8$
$\boxed{y = 4x - 3}$

2. $f(x)=x^2+x; \underline{a=-1}$

Eq of tangent: $y-\underline{f(a)}=\underline{f'(a)}(x-\underline{a})$

$$f(-1+h)=(-1+h)^2+(-1+h)$$
$$=1-2h+h^2-1+h$$
$$=h^2-h$$

$$f(-1)=(-1)^2+(-1)=1-1=0$$

$$f'(-1)=\lim_{h\to 0}\frac{h^2-h-0}{h}=\lim_{h\to 0}\frac{h(h-1)}{h}=\lim_{h\to 0}(h-1)=0-1=-1$$

$$y-0=-1(x-(-1))$$
$$y=-1(x+1)$$
$$\boxed{y=-x-1}$$

Example: If a stone is thrown down at 120 feet per second from a height of 1000 feet, its height after t seconds is given by $s(t)=1000-120t-16t^2$. Find its instantaneous velocity function and its velocity at time $t=3$.

derivative function

$v(3)$

$$v(t)=s'(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}=\lim_{h\to 0}\frac{h(-120-32t-16h)}{h}$$
$$=\lim_{h\to 0}(-120-32t-16h)$$
$$=-120-32t-16(0)$$
$$=-120-32t$$
$$* \boxed{v(t)=-120-32t} *$$

$$s(t+h)=1000-120(t+h)-16(t+h)^2$$
$$=1000-120t-120h-16(t^2+2th+h^2)$$
$$=1000-120t-120h-16t^3-32th-16h^2$$
$$s(t+h)-s(t)=-120h-32th-16h^2$$

$$v(3)=-120-32(3)$$
$$=-120-96$$
$$=-216 \text{ ft/sec}$$