11.4 The Chain Rule

Extra Practice: The Composition Handout

Chain Rule If *f* is a differentiable function of *u*, and *u* is a differentiable function of *x*, then the composite *f*(*u*) is a differentiable function of *x*, and $\frac{d}{dx}\left[f(u)\right]=f'(u)\frac{du}{dx}$. In words, the derivative of *f(something)* is the derivative of *f*, evaluated at '*something*', times the derivative of '*something*'. Or, if *f* is the outside and *u* is the inside function, then the chain rule says derivative of the outside, leave inside alone, times derivative of the inside. In differential notation: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$.

Examples: Find the derivative.

1. $f(x)=(3x-1)^2$

   Inside $= 3x-1$

   outside $= (\ )^2$

   $f'(x)=2(3x-1)^{2-1}(3) = 6(3x-1) = 18x-6$

2. $f(x)=(1-x)^{-1}$

   inside $= 1-x$

   outside $= (\ )^{-1}$

   $f'(x)=-1(1-x)^{-1-1}(-1) = 1(1-x)^{-2}$ or $(1-x)^{-2}$

3. $f(x)=(4x+7)^{-2}$

   inside $= 4x+7$

   outside $= (\ )^{-2}$

   $f'(x)=-2(4x+7)^{-2-1}(4) = -8(4x+7)^{-3}$

4. $g(x)=\frac{1}{(x^2-5x)^3}$

   Rewrite: $g(x)=(x^2-5x)^{-3}$

   inside $= x^2-5x$

   outside $= (\ )^{-3}$

   $g'(x)=-3(x^2-5x)^{-3-1}(2x-5) = -3(2x-5)(x^2-5x)^{-4}$

   Rewrite: $g'(x)=\frac{-3(2x-5)}{(x^2-5x)^4}$

5. $g(x)=\frac{3}{(7x-9)^5}$

Rewrite: $g(x) = 3(7x-9)^{-5}$

inside $= 7x-9$

outside $= (\quad)^{-5}$

$g'(x) = 3\left[-5(7x-9)^{-5-1}(7)\right] = -105(7x-9)^{-6}$

Rewrite: $g'(x) = \frac{-105}{(7x-9)^6}$

6. $r(x)=(0.1x-4.2x^{-1})^{0.5}$

inside $= 0.1x - 4.2x^{-1}$

outside $= (\quad)^{0.5}$

$r'(x) = 0.5(0.1x-4.2x^{-1})^{0.5-1}(0.1-4.2(-1x^{-1-1}))$

$r'(x) = 0.5(0.1x-4.2x^{-1})^{-0.5}(0.1+4.2x^{-2})$

7. $s(x)=\left(\frac{3x-9}{2x+4}\right)^3$

inside $= \frac{3x-9}{2x+4}$

derivative inside $= \frac{(2x+4)(3)-(3x-9)(2)}{(2x+4)^2} = \frac{6x+12-6x+18}{(2x+4)^2} = \frac{30}{(2x+4)^2}$

outside $= (\quad)^3$

$s'(x) = 3\left(\frac{3x-9}{2x+4}\right)^{3-1}\left(\frac{30}{(2x+4)^2}\right) = 3\left(\frac{3x-9}{2x+4}\right)^2\left(\frac{30}{(2x+4)^2}\right)$

yes, webassign will take this!

8. $g(x)=\underbrace{(x^2+3x)^{-2}}_{1^{st}}\underbrace{(5x^3+2)}_{2^{nd}}$ Product Rule

$1^{st} = (x^2+3x)^{-2}$ der. of $1^{st} = -2(x^2+3x)^{-3}(2x+3)$

in $= x^2+3x$

out $= (\quad)^{-2}$

$2^{nd} = (5x^3+2)$ der of $2^{nd} = 15x^2$

der of $1^{st} = -2(x^2+3x)^{-2-1}(2x+3)$

Using product rule we find

$g'(x) = \underbrace{(x^2+3x)^{-2}}_{1^{st}} * \underbrace{(15x^2)}_{\text{der } 2^{nd}} + \underbrace{(5x^3+2)}_{2^{nd}} * \underbrace{(-2(x^2+3x)^{-3}(2x+3))}_{\text{derivative of } 1^{st}}$

used chain rule to find this

Example: Paramount Electronics has an annual profit given by $P = -100,000 + 5000q - 0.25q^2$ where q is the number of laptop computers it sells each year. The number of laptop computers it can make and sell each year depends on the number n of electrical engineers Paramount employs, according to the equation $q = 30n + 0.1n^2$. Use the chain rule to find $\left.\frac{dP}{dn}\right|_{n=10}$ and interpret the result.

According to the chain rule, $\frac{dP}{dn} = \frac{dP}{dq} \cdot \frac{dq}{dn}$.

$$\frac{dP}{dq} = \frac{d}{dq}\left(-100,000 + 5000q - 0.25q^2\right) = 0 + 5000 - 0.25(2q) = 5000 - 0.5q$$

$$\frac{dq}{dn} = \frac{d}{dn}\left(30n + 0.1n^2\right) = 30 + 0.1(2n) = 30 + 0.2n$$

So $\frac{dP}{dn} = \frac{dP}{dq} \cdot \frac{dq}{dn} = (5000 - 0.5q)(30 + 0.2n)$

What about $n = 10$? We can substitute for n, but what is q?

Use the original equation to find $q(10) = 30(10) + 0.1(10)^2 = 300 + 10 = 310$.

Now $\left.\frac{dP}{dn}\right|_{n=10} = (5000 - 0.5(310))(30 + 0.2(10)) = (5000 - 155)(30 + 2) = 4845(32)$
$= 155040$

$\frac{dP}{dn}$ is change in profit per engineer. $\left.\frac{dP}{dn}\right|_{n=10} = 155,040$ means for 10 engineers, the company's profits will increase \$155,040 per electrical engineer.