11.6 Implicit Differentiation

We are used to explicit functions where *y* is a direct function of *x*. Now we will talk about implicit functions where we can't solve for *y* but we always keep in mind that *y* is a function of *x*.

Examples: Find dy/dx using implicit differentiation and by solving for *y*.

1. $4x - 5y = 9$

implicit

$$4 - 5\frac{dy}{dx} = 0$$

$$-5\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = \frac{-4}{-5} = \frac{4}{5}$$

explicit: Solve for $y \rightarrow$ $-5y = 9 - 4x$

$$y = \frac{9}{-5} - \frac{4x}{-5} \Rightarrow y = \frac{4}{5}x - 9$$

Derivative $\rightarrow$ $\frac{dy}{dx} = \frac{4}{5} + 0 = \frac{4}{5}$

2. $x - y = xy$

implicit

$$1 - \frac{dy}{dx} = 1(y) + x\frac{dy}{dx}$$

$$-y + \frac{dy}{dx} \qquad -y \qquad + \frac{dy}{dx}$$

$$1 - y = x\frac{dy}{dx} + \frac{dy}{dx}$$

$$1 - y = \frac{dy}{dx}(x+1)$$

$$\boxed{\frac{1-y}{x+1} = \frac{dy}{dx}}$$ implicit

explicit: solve for $y \rightarrow$ $x = xy + y$

$$x = y(x+1)$$

$$\frac{x}{x+1} = y$$

$$\frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{dy}{dx}$$ $\leftarrow$ derivative

$$\frac{x+1-x}{(x+1)^2} = \boxed{\frac{1}{(x+1)^2} = \frac{dy}{dx}}$$ explicit

To see that these two are the same we would need to substitute for y in the implicit derivative and simplify.

Examples: Find the derivative dy/dx using implicit differentiation.

1. $2x^2 - y^2 = 4$

$$4x - 2y\frac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{-4x}{-2y} \rightarrow \frac{dy}{dx} = \frac{2x}{y}$$

2. $\frac{xy}{2} - y^2 = 3$ $\quad \frac{1}{2}xy - y^2 = 3$

product

$$\underbrace{\frac{1}{2}(y) + \frac{1}{2}x\frac{dy}{dx}} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\frac{1}{2}x - 2y\right) = -\frac{1}{2}y \rightarrow \frac{dy}{dx} = \frac{-\frac{1}{2}y}{\left(\frac{1}{2}x - 2y\right)} \cdot \frac{2}{2} = \frac{-y}{x - 4y}$$

3. $x^2 e^y - y^2 = e^x$

product

$$2xe^y + x^2 e^y \frac{dy}{dx} - 2y\frac{dy}{dx} = e^x$$

$$\frac{dy}{dx}\left(x^2 e^y - 2y\right) = e^x - 2xe^y$$

$$\frac{dy}{dx} = \frac{e^x - 2xe^y}{x^2 e^y - 2y}$$

Examples: Use implicit differentiation to find the slope of the tangent line and the equation of the tangent line at the indicated point.

derivative at the point

1. $3x^2 - y^2 = 1 \quad (-2,1)$

A) Find $\frac{dy}{dx}$: $6x - 2y\frac{dy}{dx} = 0$

$-2y\frac{dy}{dx} = -6x$

$\frac{dy}{dx} = \frac{-6x}{-2y} = \frac{3x}{y}$

B) $\left.\frac{dy}{dx}\right|_{(-2,1)} = \frac{3(-2)}{(1)} = \frac{-6}{1} = -6$

c) Use $m=-6$ and point $(-2,1)$:

$y = mx+b$

$1 = -6(-2)+b$

$1 = 12+b$

$-11 = b$

$y = -6x - 11$

or

$y - y_1 = m(x - x_1)$

$y - 1 = -6(x+2)$

$y - 1 = -6x - 12$

$y = -6x - 11$

2. $\ln(x-y)+1 = 3x^2, \quad x = 0$

A) Find $\frac{dy}{dx}$: $\frac{1 - \frac{dy}{dx}}{x-y} + 0 = 6x$

$\frac{1}{x-y} - \frac{\frac{dy}{dx}}{x-y} = 6x$

$-(x-y) \quad \frac{-\frac{dy}{dx}}{x-y} = \left(6x - \frac{1}{x-y}\right)(-(x-y))$

$\frac{dy}{dx} = -6x(x-y) + 1$

$\ln(0-y)+1 = 3(0)^2$

$\ln(-y)+1 = 0$

$\ln(-y) = -1$

$-y = e^{-1}$

so $y = -e^{-1}$

B) $\left.\frac{dy}{dx}\right|_{(0,-e^{-1})} = -6(0)(0+e^{-1})+1 = 1$

c) equation of tangent line. $y-(-e^{-1}) = 1(x-0)$

$y + e^{1} = x$

$y = x - e^{-1}$