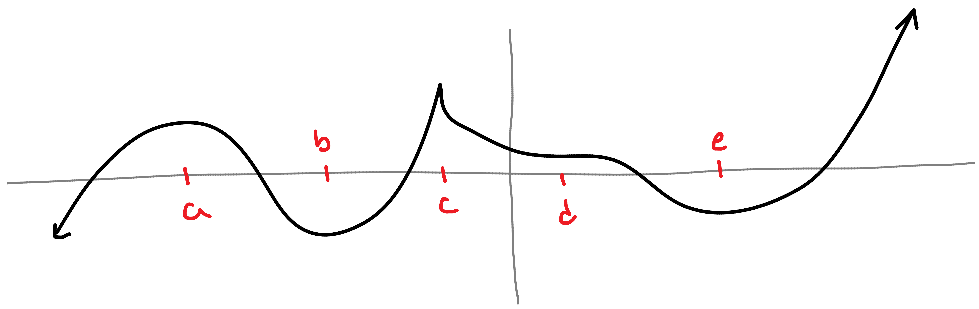
Chapter 12 Applications of the Derivative

Now you must start simplifying all your derivatives. The rule is, if you need to use it, you must simplify it.

12.1 Maxima and Minima



Relative Extrema:

> f has a relative maximum at c if there is some interval (r, s) (even a very small one) containing c for which $f(c) \geq f(x)$ for all x between r and s for which $f(x)$ is defined.
>
> f has a relative minimum at c if there is some interval (r, s) (even a very small one) containing c for which $f(c) \leq f(x)$ for all x between r and s for which $f(x)$ is defined.

Absolute Extrema

> f has an absolute maximum at c if $f(c) \geq f(x)$ for every x in the domain of f.
>
> f has an absolute minimum at c if $f(c) \leq f(x)$ for every x in the domain of f.

Extreme Value Theorem - If f is continuous on a closed interval $[a,b]$, then it will have an absolute maximum and an absolute minimum value on that interval. Each absolute extremum must occur either at an endpoint or a critical point. Therefore, the absolute max is the largest value in a table of vales of f at the endpoints and critical points, and the absolute minimum is the smallest value.

So what is a critical point?

Locating Candidates for Relative Extrema If *f* is a real valued function, then its relative extrema occur among the following types of points, collectively called *critical points*:

1. Stationary Points: *f* has a stationary point at *x* if *x* is in the domain of *f* and *f′(x) = 0*. To locate stationary points, set *f′(x) = 0* and solve for *x*.

2. Singular Points: *f* has a singular point at *x* if *x* is in the domain and *f′(x)* is not defined. To locate singular points, find values of *x* where *f′(x)* is not defined, but *f(x)* is defined.

3. Endpoints: The *x*-coordinates of endpoints are endpoints of the domain, if any. Recall that closed intervals contain endpoints, but open intervals do not.

Examples: Find the exact location of all the relative and absolute extrema of each function. Also, determine the intervals on which the function is increasing and decreasing.

1. $f(x) = 2x^2 - 2x + 3$ with domain $[0,3]$

$f'(x) = 4x - 2$

Stationary pts: $0 = 4x - 2$
$2 = 4x$
$\frac{1}{2} = \frac{2}{4} = x$

Singular pts: NONE

Endpoints: $x = 0$
$x = 3$

always use original f(x) to find y-values

absolutes

$f(0) = 3$ relative max
$f(\frac{1}{2}) = 2(\frac{1}{2})^2 - 2(\frac{1}{2}) + 3 = 2.5$ ← minimum
$f(3) = 2(3)^2 - 2(3) + 3 = 15$ ← maximum

It seems that f is decreasing for x-values $(0, \frac{1}{2})$ and increasing for x-values $(\frac{1}{2}, 3)$

2. $f(x) = 2x^3 - 6x + 3$ with domain $[-2,2]$

$f'(x) = 6x^2 - 6$

Stationary pts: $0 = 6x^2 - 6$
$0 = 6(x^2 - 1)$
$0 = x^2 - 1$
$1 = x^2$
$\pm 1 = \pm\sqrt{1} = x$

Singular: NONE

Endpoints: $x = -2$
$x = 2$

$f(-2) = -1$ minimum
$f(-1) = 7$ maximum
$f(1) = -1$ minimum
$f(2) = 7$ maximum

↑ all absolutes

Increasing $(-2,-1)$
decreasing $(-1,1)$
Increasing $(1,2)$

First Derivative Test for Extrema – Suppose that c is a critical point of the continuous function f, and that its derivative is defined for x close to, and on both sides of, $x = c$. Then, determine the sign of the derivative to the left and right of $x = c$.

1. If $f'(x)$ is positive to the left and negative to the right, then f has a maximum at $x = c$.
2. If $f'(x)$ is negative to the left and positive to the right, then f has a minimum at $x = c$.
3. If $f'(x)$ has the same sign on both sides of $x = c$, then f has neither a maximum nor a minimum at $x = c$.

Now we can use a sign chart to test for sure where a function is increasing and where it is decreasing.

Previous Examples Continued –

3. $f(t) = t^3 - 3t^2$ with domain $[-1, \infty)$ *

$f'(t) = 3t^2 - 6t$

$0 = 3t^2 - 6t$

$0 = 3t(t-2)$

$3t = 0$ or $t - 2 = 0$

$t = 0 \qquad t = 2$

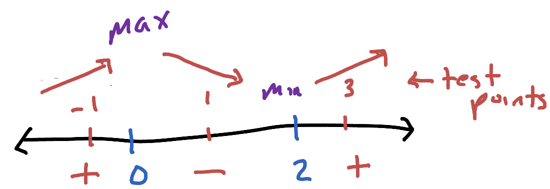
No singular points

endpoint at $x = -1$

$f(-1) = -4$ min

$f(0) = 0$ max

$f(2) = -4$ min



$f'(-1) = 3(-1)(-1-2) = (-3)(-3) =$ positive

$f'(1) = 3(1)(1-2) = 3(-1) =$ negative

$f'(3) = 3(3)(3-2) = 9(1) =$ positive

Increasing: $(-1, 0)$ *, $(2, \infty)$ Keep endpoints in mind

decreasing: $(0, 2)$

4. $f(x) = (x+1)^{2/5}$ with domain $[-2, 0]$

$f'(x) = \frac{2}{5}(x+1)^{-3/5}(1) = \dfrac{2}{5\sqrt[5]{(x+1)^3}}$

$f'(x) = 0$ never

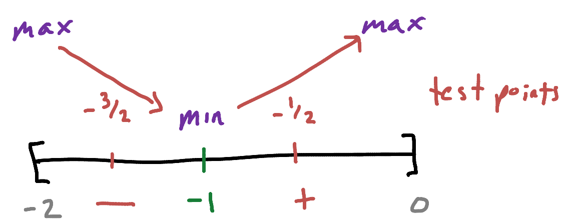
$f'(x)$ is undefined at $x = -1$

endpoints $x = -2$, $x = 0$

max $f(-2) = (-2+1)^{2/5} = (-1)^{2/5} = \sqrt[5]{(-1)^2} = 1$

min $f(-1) = (-1+1)^{2/5} = 0$

max $f(0) = (0+1)^{2/5} = 1$



$f'(-\frac{3}{2}) = \dfrac{2}{5\sqrt[5]{(-\frac{3}{2}+1)^3}} = \dfrac{\text{positive}}{\text{negative}} = -$

$f'(-\frac{1}{2}) = \dfrac{2}{5\sqrt[5]{(-\frac{1}{2}+1)^3}} = \dfrac{\text{pos}}{\text{pos}} = +$

increasing: $(-1, 0)$

decreasing: $(-2, -1)$

5. $f(t)=\dfrac{t^2+1}{t^2-1}$ with domain $[-2,2]$

not defined at $t=\pm 1$

$$f'(t)=\frac{(t^2-1)(2t)-(t^2+1)(2t)}{(t^2-1)^2}=\frac{\cancel{2t^3}-2t-\cancel{2t^3}-2t}{(t^2-1)^2}=\frac{-4t}{(t^2-1)^2}$$

$0=f'(t)$ is $0=-4t$
$0=t$

$f'(t)$ undefined at $t=\pm1$, not singular pts

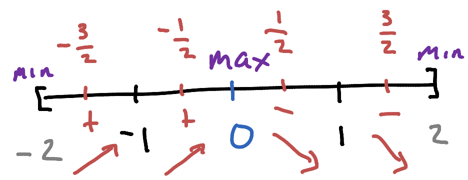
endpoints $t=-2$, $t=2$

$f(-2)=\frac{5}{3}$ min

$f(0)=\frac{1}{-1}=-1$ max

$f(2)=\frac{5}{3}$ min

weird but true



Notice that for $f'(t)=\dfrac{-4t}{(t^2-1)^2}$, the denominator is always positive.

$f'(-\frac{3}{2})=\dfrac{-4(-\frac{3}{2})}{pos}=\dfrac{pos}{pos}=+$ $\qquad f'(\frac{1}{2})=\dfrac{neg}{pos}=-$

$f'(-\frac{1}{2})=\dfrac{-4(-\frac{1}{2})}{pos}=\dfrac{pos}{pos}=+$ $\qquad f'(\frac{3}{2})=\dfrac{neg}{pos}=-$

Increasing $(-2,-1)$, $(-1,0)$

decreasing $(0,1)$, $(1,2)$