12.2 Applications of Maxima and Minima

When dealing with costs, we would like to know a minimum whereas with profit we always want to maximize. These are called optimal values because they are the best possible case for the problem at hand.

In all applications the first step is to translate a written description into a mathematical problem. In the problems we look at in this section, there are unknowns that we are asked to find, there is an expression involving those unknowns that must be made as large or small as possible - the objective function - and there may be constraints - equations or inequalities relating the variables.

Solving Optimization Problems

1. Identify the unknowns, possibly with the aid of a diagram.
2. Identify the objective function.
3. Identify the constraint equations.
4. State the optimization problem.
5. Eliminate extra variables.
6. Find absolute min (or max) of the objective function.

Skills Examples: Solve the optimization problems.

1. Maximize $P = xy$ with $x + y = 10$.

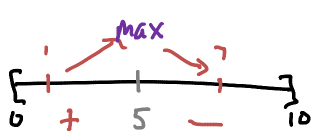
objective function $P = xy$ has two variables. Let's try to substitute for one of them

Constraint: $x + y = 10$ or $y = 10 - x$. Substitute this into objective function

$P = x(10 - x) = 10x - x^2$ max occurs at critical point so find P'

$P' = 10 - 2x$ stationary pt: $0 = 10 - 2x$, $2x = 10$, $x = 5$ no singular points

reasonable endpoints $x = 0$, $x = 10$



$P'(1) = 10 - 2(1) = 10 - 2 = 8 =$ positive

$P'(7) = 10 - 2(7) = 10 - 14 =$ negative

maximum of P when $x = 5$ and $y = 10 - 5 = 5$

2. Minimize $F = x^2 + y^2$ with $x + 2y = 10$.

$X = 10 - 2y$

$F = (10 - 2y)^2 + y^2$

derivative $F' = 2(10-2y)(-2) + 2y$

Simplify $F' = -4(10-2y) + 2y$

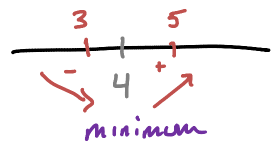
$F' = -40 + 8y + 2y$

$F' = 10y - 40$

$F' = 0 = 10y - 40$

$40 = 10y$

$4 = y$



$F'(3) = 10(3) - 40 =$ negative

$F'(5) = 10(5) - 40 =$ positive

F has a minimum at $y = 4$ and $X = 10 - 2(4) = 2$.

Application Examples:

1. Assume that is costs Microsoft approximately $C(x) = 14,400 + 550x + 0.01x^2$ dollars to manufacture *x* Xbox 360s in a day. How many Xboxes should be manufactured in order to minimize average cost? What is the resulting average cost of an Xbox? Give your answer to the nearest dollar.

$\bar{C}(x) = \frac{C(x)}{x}$, objective function

$$\bar{C}(x) = \frac{14400 + 550x + 0.01x^2}{x} = \frac{14400}{x} + \frac{550x}{x} + \frac{0.01x^2}{x} = 14400x^{-1} + 550 + 0.01x$$

$$\bar{C}'(x) = -14400x^{-2} + 0 + 0.01 = \frac{-14400}{x^2} + 0.01$$

$\bar{C}'(x)$ undefined at $x = 0$, singular point.

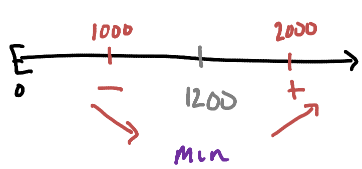
$\bar{C}'(x) = 0$ when $0 = \frac{-14400}{x^2} + 0.01 \longrightarrow \cancel{x^2}(-0.01) = \frac{-14400}{\cancel{x^2}} \cdot x^2 \rightarrow -0.01x^2 = -14400$

$x^2 = \frac{-14400}{-0.01}$

$x^2 = 1440000$

$x = \pm 1200$

$x = -1200$ unreasonable in this situation



$\bar{C}'(1000) = \frac{-14400}{(1000)^2} + 0.01 = -0.0044$

$\bar{C}'(2000) = \frac{-14400}{(2000)^2} + 0.01 = 0.0064$

Minimum average cost when $x = 1200$ xBoxs are manufactured.

Resulting average cost per xBox is $\bar{C}(1200) = \frac{14400}{1200} + 550 + 0.01(1200) = \$574/\text{xBox}$

Note: This is why prices are so high for initial release of a product.

2. Hercules Films is deciding on the price of the video release of its film Bride of the Son of Frankenstein. Marketing estimates that at a price of p dollars, it can sell $q = 200,000 - 10,000p$ copies, but each copy cost \$4 to make. What price will give the greatest profit?

$C = 4q = 4(200,000 - 10,000p) = 800,000 - 40,000p$

$R = pq = p(200,000 - 10,000p) = 200,000p - 10,000p^2$

constraints

$\text{Profit} = R - C$

$\text{Profit} = 200,000p - 10,000p^2 - (800,000 - 40,000p)$

$= 200,000p - 10,000p^2 - 800,000 + 40,000p$

$= 240,000p - 10,000p^2 - 800,000$ objective function

$\dfrac{d\,\text{Profit}}{d\,\text{price}} = 240,000 - 20,000p - 0$

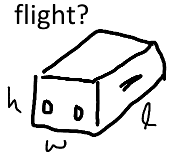
derivative $= 0$ is $0 = 240,000 - 20,000p$

$20,000p = 240,000$

$p = \dfrac{240,000}{20,000} = \$12$

At a price of \$12 per video the company will maximize profit.

3. American Airlines requires that the total outside dimensions (length + width + height) of a checked bag not exceed 62 inches. Suppose you want to check a bag whose height is equal to its width. What is the largest volume bag of this shape that you can check on an American Airlines flight?



Constraints: $l + w + h \le 62$ and $h = w$

objective: $V = lwh$ largest $\Rightarrow$ maximum

Using constraints together and equality instead of inequality:

$l + w + h = 62$ becomes $l + w + w = 62$ or $l + 2w = 62$ so $l = 62 - 2w$

Substitute $V = lwh = (62 - 2w)ww = (62 - 2w)w^2 = 62w^2 - 2w^3$

maximize $\Rightarrow V' = 124w - 6w^2$

$0 = 2w(62 - 3w)$

$2w = 0$ or $62 - 3w = 0$

$w = 0$ or $62 = 3w$

not a max

$\dfrac{62}{3} = w$ must be max check to be sure

If $w = \frac{62}{3} = 20\frac{2}{3}$ inches

$h = w = 20\frac{2}{3}$ inches

and $l = 62 - 2\left(\frac{62}{3}\right) = 20\frac{2}{3}$ in.

The max volume will be cube shaped

4. The demand, in rides per day, for monorail service in Las Vegas in 2005 can be approximated by $q = -4{,}500p + 41{,}500$ when the fare was $\$p$. What price should have been charged to maximize total revenue?

$R = pq = p(-4500p + 41500) = -4500p^2 + 41500p$

$R' = -9000p + 41500$

$0 = -9000p + 41500$

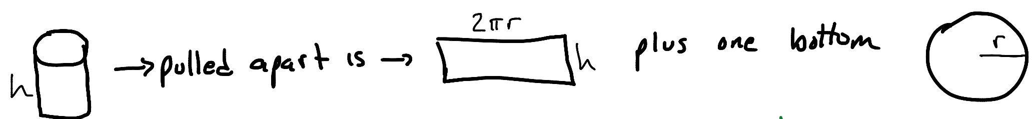
$9000p = 41500$

$p = \frac{41500}{9000} = 4.6\overline{1}$

They should have charged \$4.61

5. A company manufactures cylindrical metal drums with open tops with a volume of 1 cubic meter. What should be the dimensions of the drum in order to use the least amount of metal in their production?

Surface area



Constraint $V = 1 = \pi r^2 h$ (volume of a cylinder)

use $V = 1 = \pi r^2 h$ to find $\frac{1}{\pi r^2} = h$

objective $SA = \underbrace{2\pi r h}_{\text{area of } \square} + \underbrace{\pi r^2}_{\text{area of } \bigcirc}$

$SA = 2\pi r\left(\frac{1}{\pi r^2}\right) + \pi r^2$

$SA = \frac{2}{r} + \pi r^2 = 2r^{-1} + \pi r^2$

singular point at $r=0$

derivative $SA' = -2r^{-2} + 2\pi r = -\frac{2}{r^2} + 2\pi r$

$SA' = 0 \Rightarrow \quad 0 = -\frac{2}{r^2} + 2\pi r$

$r^2 \cdot \frac{2}{r^2} = 2\pi r \cdot r^2$

$2 = 2\pi r^3$

$\frac{2}{2\pi} = r^3$

$\frac{1}{\pi} = r^3$ so $r = \sqrt[3]{\frac{1}{\pi}} = \left(\frac{1}{\pi}\right)^{(1/3)} \approx 0.6828$ m

The radius should be about 0.683 meters and height

$h = \frac{1}{\pi\left(\sqrt[3]{\frac{1}{\pi}}\right)^2} \approx 0.47$ meters.