12.3 Higher Order Derivatives: Acceleration and Concavity

Definition: If a function *f* has a derivative that is in turn differentiable, then its second derivative is the derivative of the derivative of *f*, written as *f''*. If *f''*(*a*) exists, we say that *f* is twice differentiable. The acceleration of a moving object is the derivative of its velocity - that is, the second derivative of its position function.

If s = position,
$s' = v$ = velocity
$s'' = v' = a$ = acceleration

Notation: $f''(x) = \dfrac{d^2 f}{dx^2}$

Examples: Find the acceleration function and the acceleration at the given time *t*.

1. $s = -12 + t - 16t^2;\ t = 2$

$s' = V = 0 + 1 - 32t$

$v' = a = -32$

acceleration at $t = 2$ is $a(2) = -32$

This is constant acceleration.

2. $s = \sqrt{t} + t^2;\ t = 4$

$v = \frac{1}{2}t^{-1/2} + 2t$

$a = \frac{1}{2}\left(-\frac{1}{2}t^{-3/2}\right) + 2 = \dfrac{-1}{4\sqrt{t^3}} + 2$

acceleration at $t = 4$

$a(4) = \dfrac{-1}{4\sqrt{(4)^3}} + 2 = \dfrac{-1}{4\sqrt{64}} + 2 = \dfrac{-1}{4(8)} + 2$
$= -\frac{1}{32} + 2$
$= 1.96875$

Definition A curve is concave up if its slope is increasing, in which case the second derivative is positive. A curve is concave down if its slope is decreasing, in which case the second derivative is negative. A point where the graph of *f* changes concavity is called a *point of inflection*. At a point of inflection, the second derivative is zero or undefined.

Locating Points of Inflection - To locate possible points of inflection, list points where *f''(x) = 0* and also points where *f''(x)* is not defined.

The point of inflection where it changes from concave up to concave down is called the point of diminishing returns. This means that even though sales (or profit) continue to rise, the rate at which they rise is decreasing.

The Second Derivative Test for Relative Extrema - Suppose that the function f has a stationary point at $x=c$, and that $f''(c)$ exists. Determine the sign of $f''(c)$.

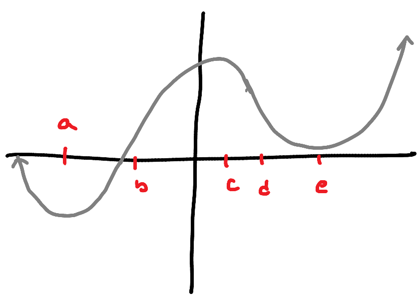
1. If $f''(c) > 0$ then f has a relative minimum at $x = c$.

2. If $f''(c) < 0$ then f has a relative maximum at $x = c$.

If $f''(c) = 0$ then the test is inconclusive and you need to use one of the methods of section 12.2 (1st derivative test).

Examples In the following graphs, use your knowledge of the relationships between *f, f', and f''* to find stationary points, increasing/decreasing, points of inflection and concavity.

① The graph of f

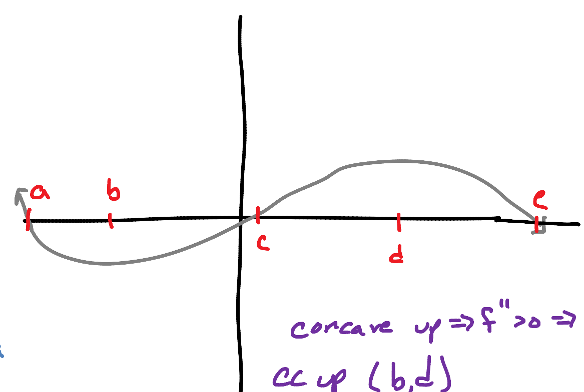


stationary points at $f'=0$, slope of tangent is zero
so at $x = a, c, e$

increasing: (a,c), (e,∞)
decreasing. $(-\infty,a)$, (c,e)
points of inflection at $x = b, d$
Concave up: $(-\infty, b)$, (d,∞)
Concave down: (b,d)

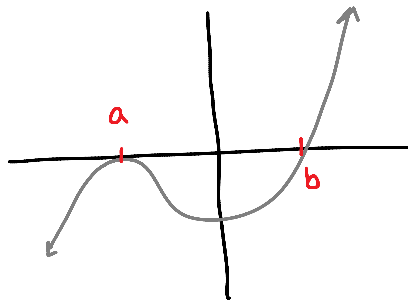
② The graph of f'
(the first derivative)

$f'=0$ at x-intercepts so
at $x=a,c,e$ are extrema
f increasing when f' is positive
increasing $(-\infty,a)$, (c,e)
decreasing (a,c) (e,∞)
point of inflection at $f''=0$ so f' extrema
P.O.I. at $x=b,d$



concave up $\Rightarrow f''>0 \Rightarrow f'$ increases
cc up (b,d)
cc down $(-\infty,b)$, (d,∞)

③ The graph is of f''
(the second derivative)



we are unable to find extrema for f given only f''

we are unable to find increasing/decreasing for f given only f''

points of inflection of f when $f''=0$
so at $x=a, b$

Concavity of f depends on positive/negative of f''
concave up (b,∞)
concave down $(-\infty,a)$, (a,b)

Examples: Find critical points and determine max/min using 2nd derivative test.

1. $f(x)=2x^2-2x+3$

$f'(x)=4x-2$

$f''(x)=4$

critical points at $0=4x-2$
$2=4x$
$\frac{1}{2}=\frac{2}{4}=x$

f'' is always positive, so always concave up, so $x=\frac{1}{2}$ is a minimum.

2. $g(x)=2x^3-6x+3$

$g'(x)=6x^2-6$

$g''(x)=12x$

$g'(x)=0$ is $0=6x^2-6$
$6=6x^2$
$1=x^2$
$\pm 1=x$ critical points

$g''(-1)=12(-1)=-12$ ← negative ⇒ down ⇒ maximum

$g''(1)=12(1)=12$ ← positive ⇒ up ⇒ minimum

3. $f(x)=3x^4-2x^3$

$f'(x)=12x^3-6x^2$

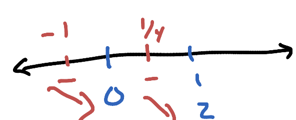
$f''(x)=36x^2-12x$

$f'(x)=0$ is $0=6x^2(2x-1)$
$6x^2=0$ or $2x-1=0$
$x=0$ or $2x=1$
$x=\frac{1}{2}$

$f''(0)=0$

$f''(\frac{1}{2})=36(\frac{1}{2})^2-12(\frac{1}{2})=36(\frac{1}{4})-6=9-6=3$ minimum

no information - use 1st derivative test



no sign change, not a max or min

$f'(-1)=-18$

$f'(\frac{1}{4})=-0.1875$

Example: The position of a particle moving in a straight line is given by $s = t^3 - t^2$ feet after *t* seconds. Find an expression for its acceleration after a time *t*. Is its velocity increasing or decreasing when *t=1*?

$V = 3t^2 - 2t$

$a = 6t - 2$

Velocity increases when acceleration is positive

Velocity decreases when acceleration is negative

$a(1) = 6(1) - 2 = 6 - 2 = 4$ a positive value

The velocity is increasing when $t = 1$