Efficiently Computing Many Roots of a Function

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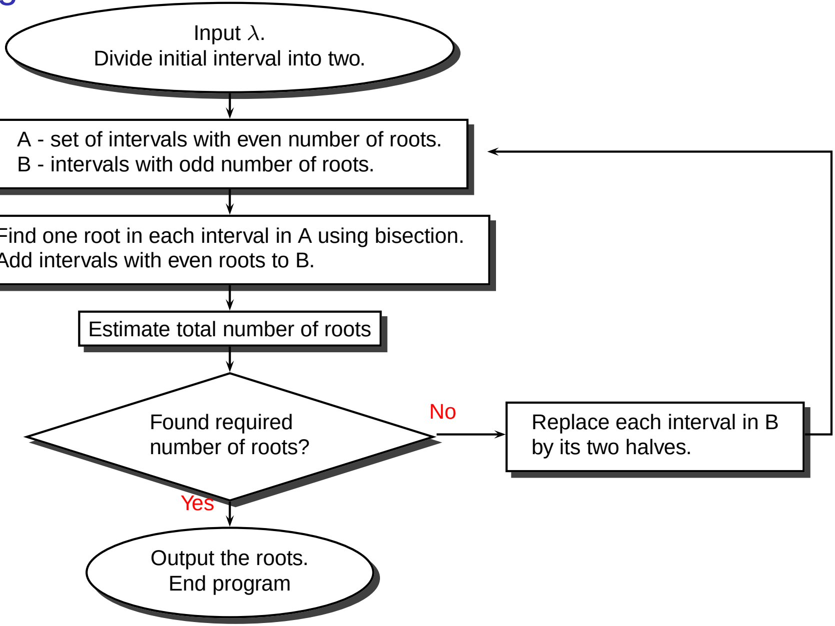
Finding Zeros of Functions

- Importance in practical applications
  Example: Finding local extrema (the zeros of a differential functions)
- Importance in theoretical problems
  Example: Riemann's hypothesis (open mathematical problem)

Strengths of Algorithm

- Simplicity - easy to implement.
- Highly efficient to reduce the problem - takes advantage of large number of roots.
- Bisection method only requires the sign of the function at a point.
- Backed by robust analysis of algorithm’s expected behavior.

Algorithm



Input λ.
Divide initial interval into two.
A - set of intervals with even number of roots.
B - intervals with odd number of roots.
Find one root in each interval in A using bisection.
Add intervals with even roots to B.
Estimate total number of roots
Found required number of roots?
No
Replace each interval in B by its two halves.
Yes
Output the roots.
End program

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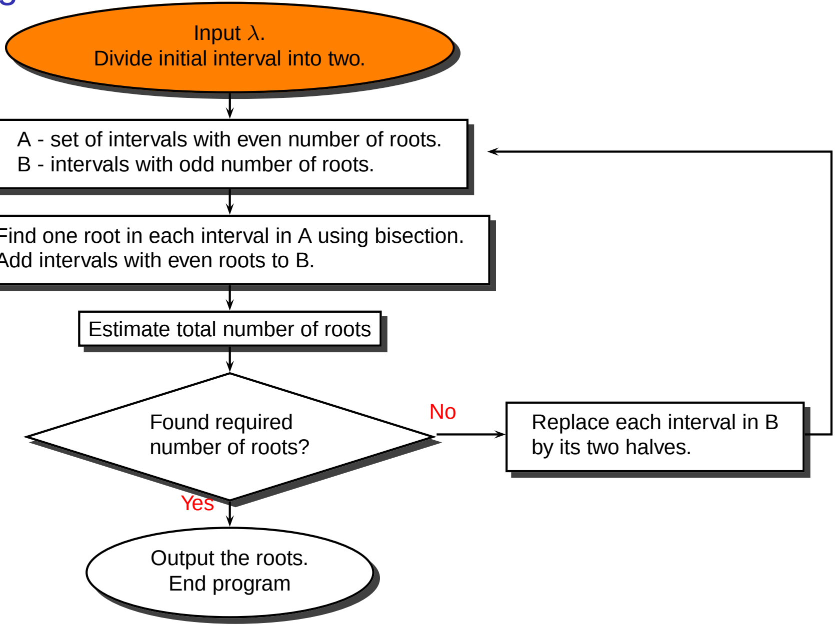
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Input λ

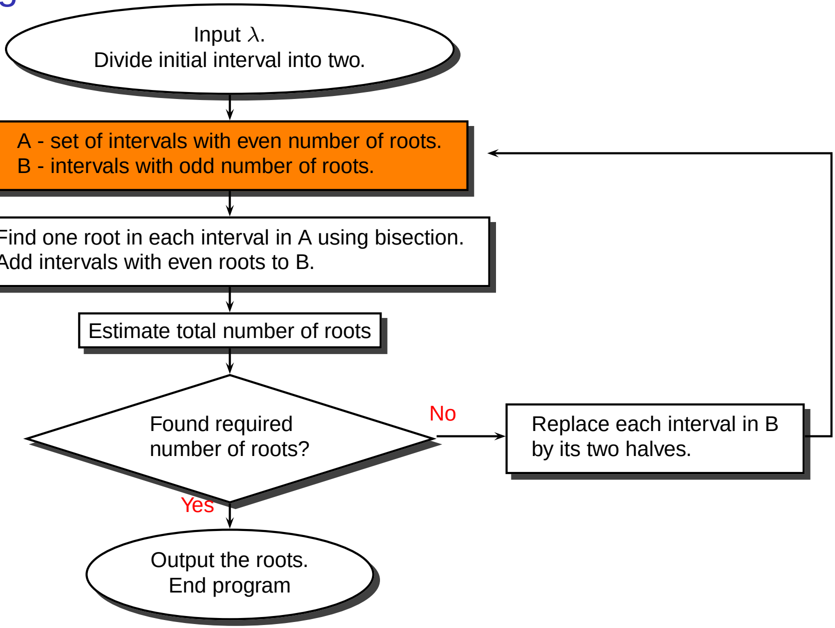
- λ is the fraction of roots to be found.
- It is shown that

$$\lambda = \frac{2^{iN} - (2^i - 2)^N}{2^{i(N-1)+1}},$$

where N is the total number of roots in the problem, i is the iteration where λ is achieved.

λ	N=100		N=500		N=1000		N=5000	
	i	w	i	w	i	w	i	w
0.5	7	776	10	3508	11	6515	13	25522
0.7	8	1093	11	5175	12	9650	14	37146
0.9	10	1919	13	11313	14	21724	16	83230
0.95	11	2898	14	19203	15	37454	17	144998

Algorithm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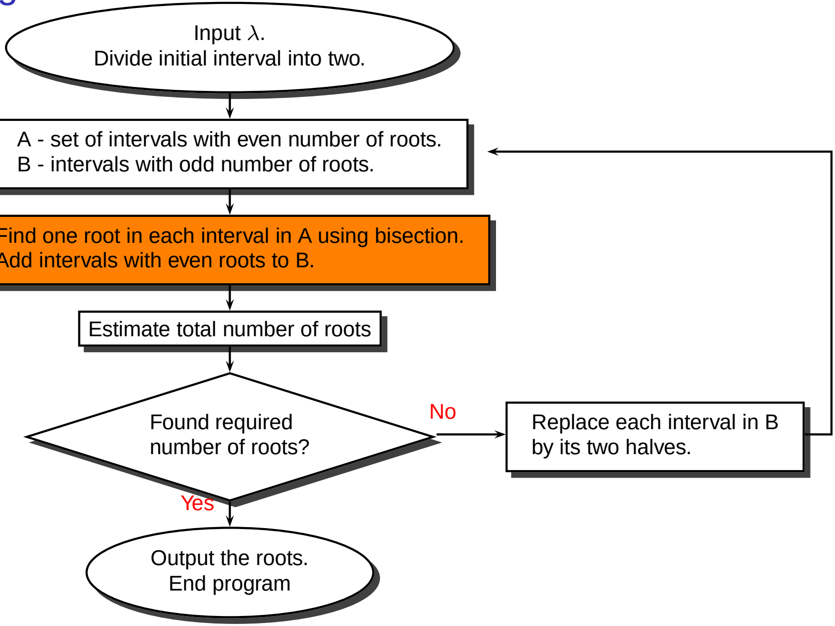
Sets of Intervals

- A is the set of intervals with an odd number of roots.
- B is the set of intervals with an even number of roots.
- Theorem giving the expected number of odd subintervals, hence the expected number of discovered roots in iteration i.

$$E_d = \frac{m^N - (m-2)^N}{2m^{(N-1)}},$$

where m is the number of equal subintervals.

Algorithm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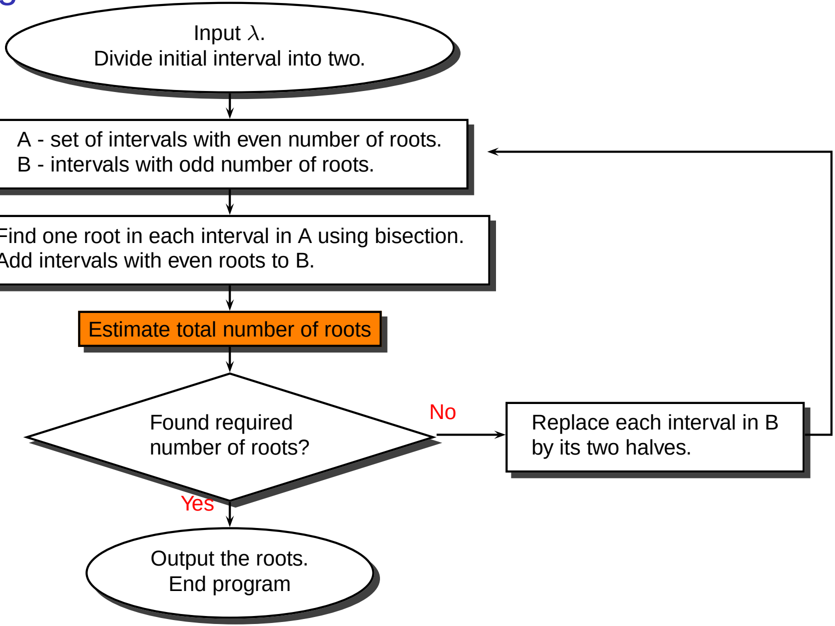
Find roots using Bisection Method

- Bisection Method always converges in interval (a, b) and is globally convergent.
- Asymptotically the best possible rate of convergence in the worst case.
- Use simplified version of Bisection Method

$$x_{i+1} = x_i + c \operatorname{sgn} f(x_l)/2^{i+1},$$

where $c = \operatorname{sgn} f(a)(b - a)$

Algorithm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Estimate total number of roots

The total number roots N is estimated by

$$N = \frac{N_{lower} + N_{upper}}{2}$$

where N_{lower} and N_{upper} are the solutions to

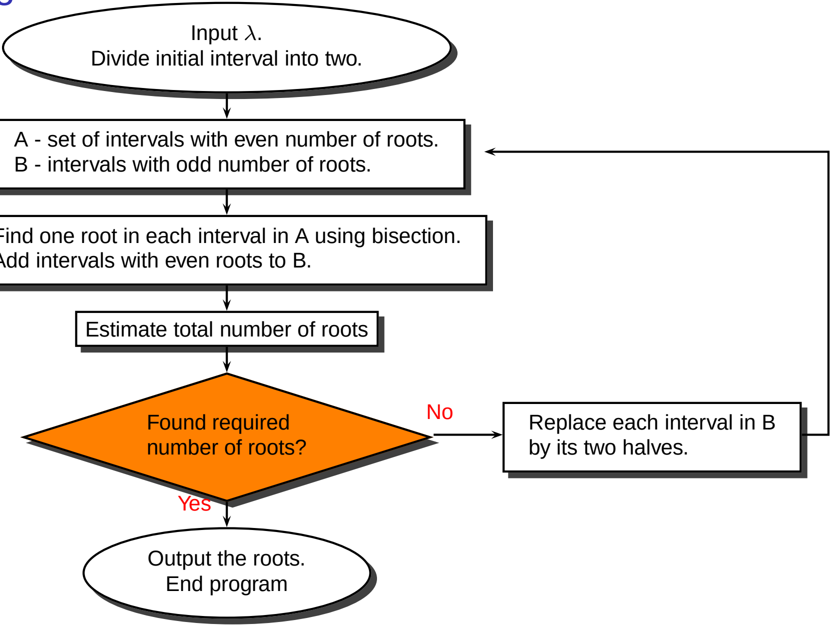
$$p_{lower} = \frac{1 - (1 - 2^{1-i})^{N_{lower}}}{2} \quad \text{and} \quad p_{upper} = \frac{1 - (1 - 2^{1-i})^{N_{upper}}}{2}$$

where

$$p_{lower} = \frac{k - z_{\alpha/2}\sqrt{\frac{k(m-k)}{m}}}{m} \quad \text{and} \quad p_{upper} = \frac{k + z_{\alpha/2}\sqrt{\frac{k(m-k)}{m}}}{m},$$

where $k = \mathrm{car}(A)$ and $z_{\alpha/2}$ is the standard normal value.

Algorithm



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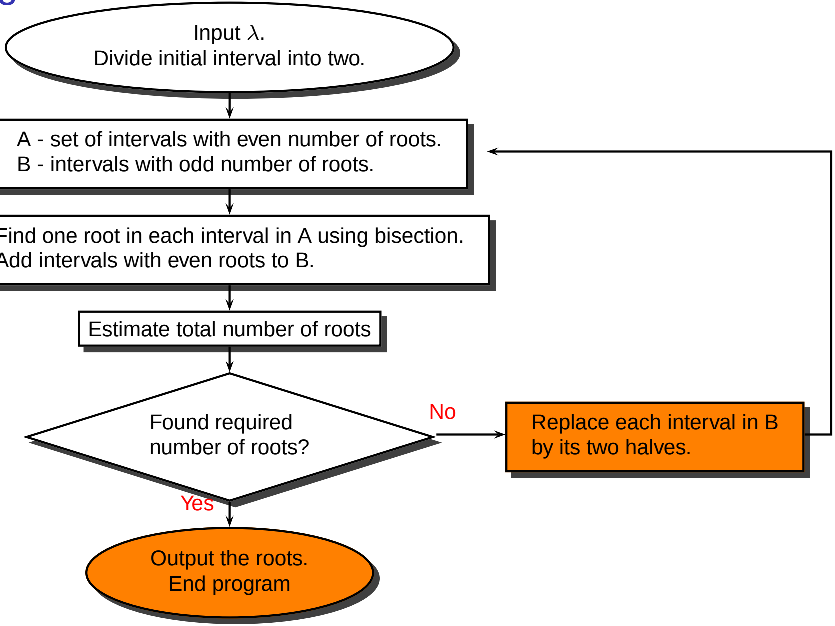
Stopping Criterion

Algorithm stops if either

- N_{lower} or N_{upper} is infinite, or
- the number of roots found is at least λN

Algorithm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Conclusion and Further Research

- Algorithm discovers roots effectively until the problem has been greatly reduced.
- Its simplicity results in fairly easy programming.
- It may be possible to extend method to higher dimensions.
- Consider an arbitrary distribution of roots.
- Apply algorithm on natural problem.