Introduction to Compressive Sampling.

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Conventional Sampling and Data Compression

- The Sampling:
  - Sample at a rate twice the highest frequency in the signal(Nyquist Theorem)
- The Compression:
  - Use appropriate basis transformation and change the huge data set into relatively fewer sample.
  - mapping images content into a vector of sparse coefficients and encode the most significant coefficient and ignore the smaller ones.
  - Examples:
    - Discrete Cosine Transform (DCT) in JPEG
    - Discret Wavelete Transform (DWT) in JPEG - 2000

problems with conventional sampling

- there are always fewer sensors
- measurements are very expensive.
- measurement process is slow.

Questions .. ?

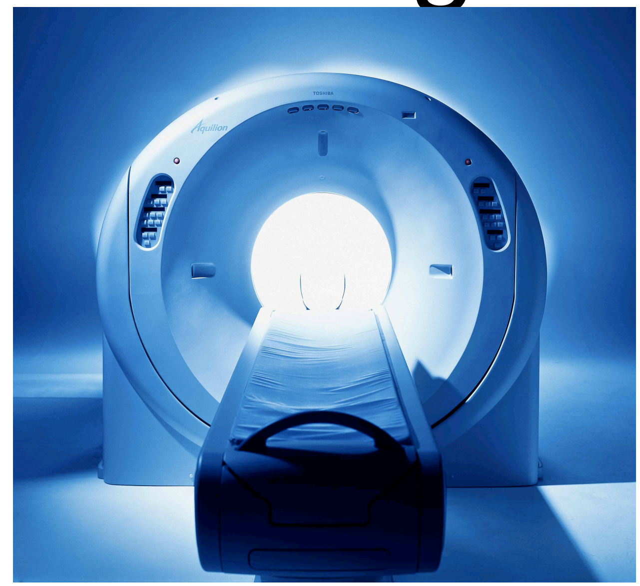
- Is it possible to do the data compression while doing the data acquisition (sampling only those samples that enable full recovery)
- consider a N – length signal $f \varepsilon R^N$

  $y_k = \langle f, \phi_k \rangle \quad k = 1, \ldots M$ where $M \ll N$
- If so , which measurement should we take?
- How should we reconstruct the original signal?

Contemporary Paradox

- Today trend:
  - acquisition of massive amount of data.
  - Most of the data will be discarded (Compression works very well).
  - Seems enormously wasteful.
- Q: Can't we just measure the part that won't end up being thrown away.

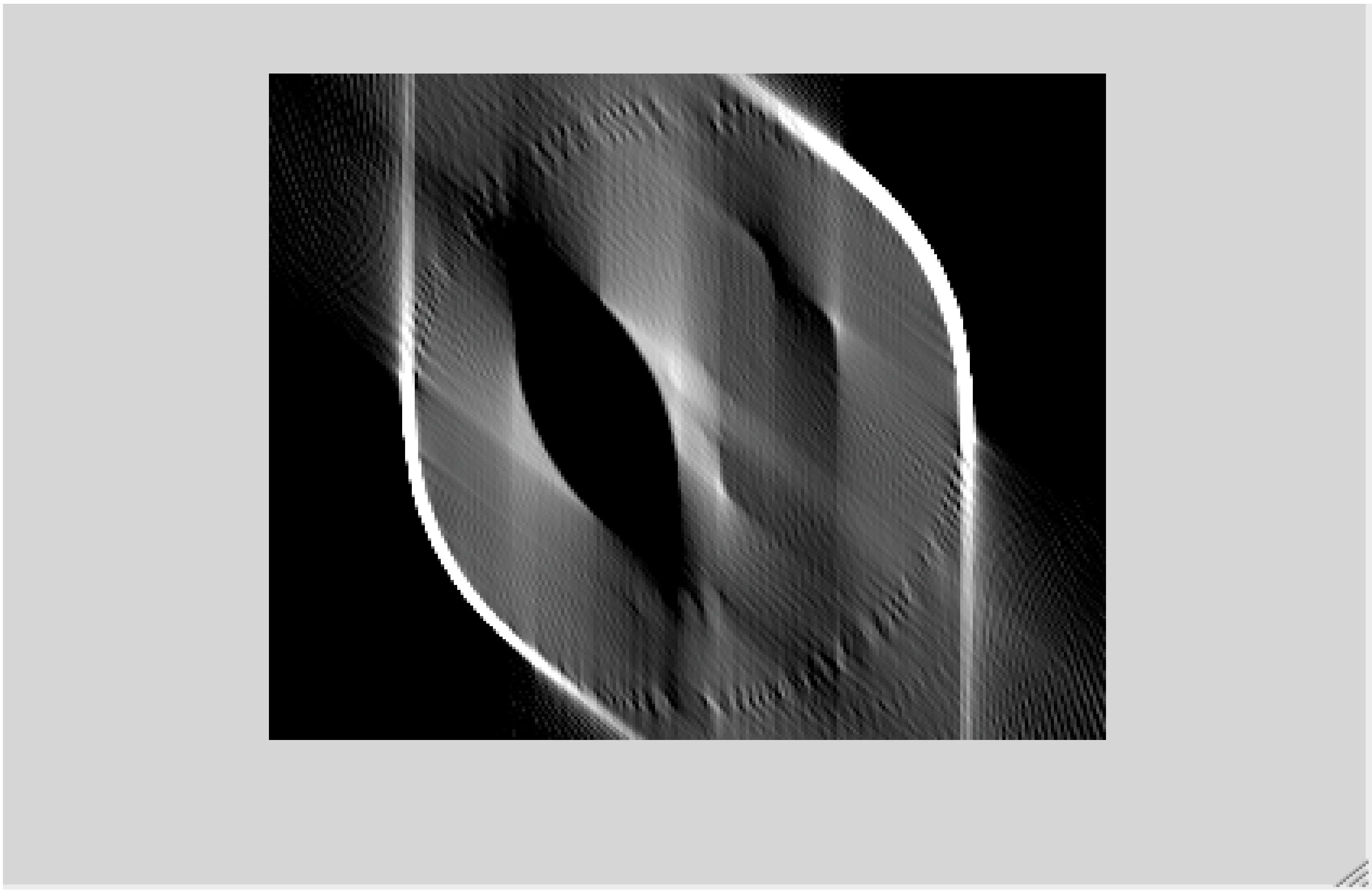
First Example: Fourier Encoding



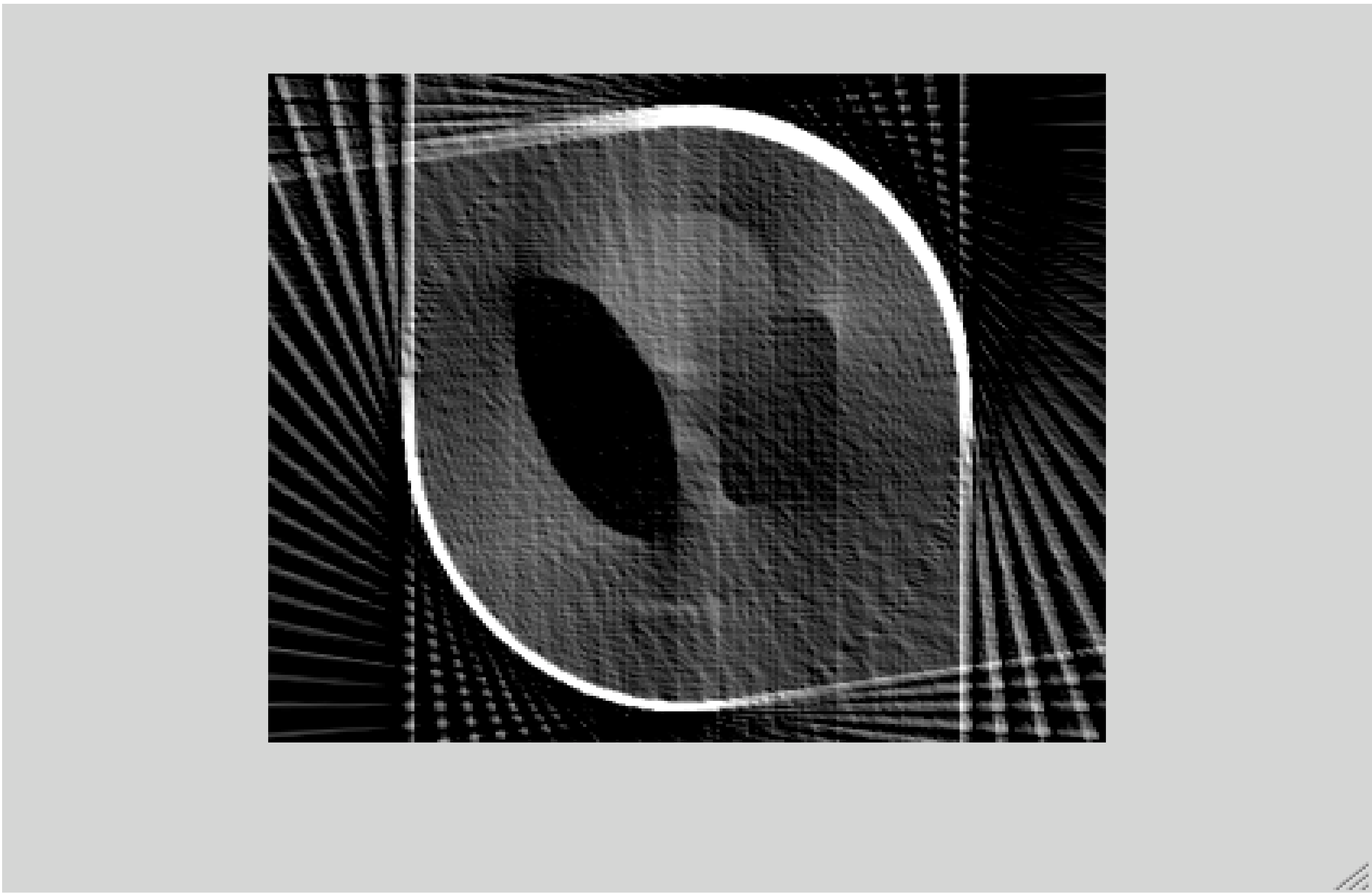
- f : 2D p x p pixle image
- $N = p^2$
- $f(\omega 1,\omega 2) = \Sigma f(t1,t2)\exp(-$
- $2\pi i(\omega 1 t1 + \omega 2 t2)$
- if we under-sample $M<<N$
- Conventional wisdom: Reconstruction impossible.
- number of samples must much number of unknowns.

Classical Reconstruction: Sheppa – Logan Head Phantom 256 X 256

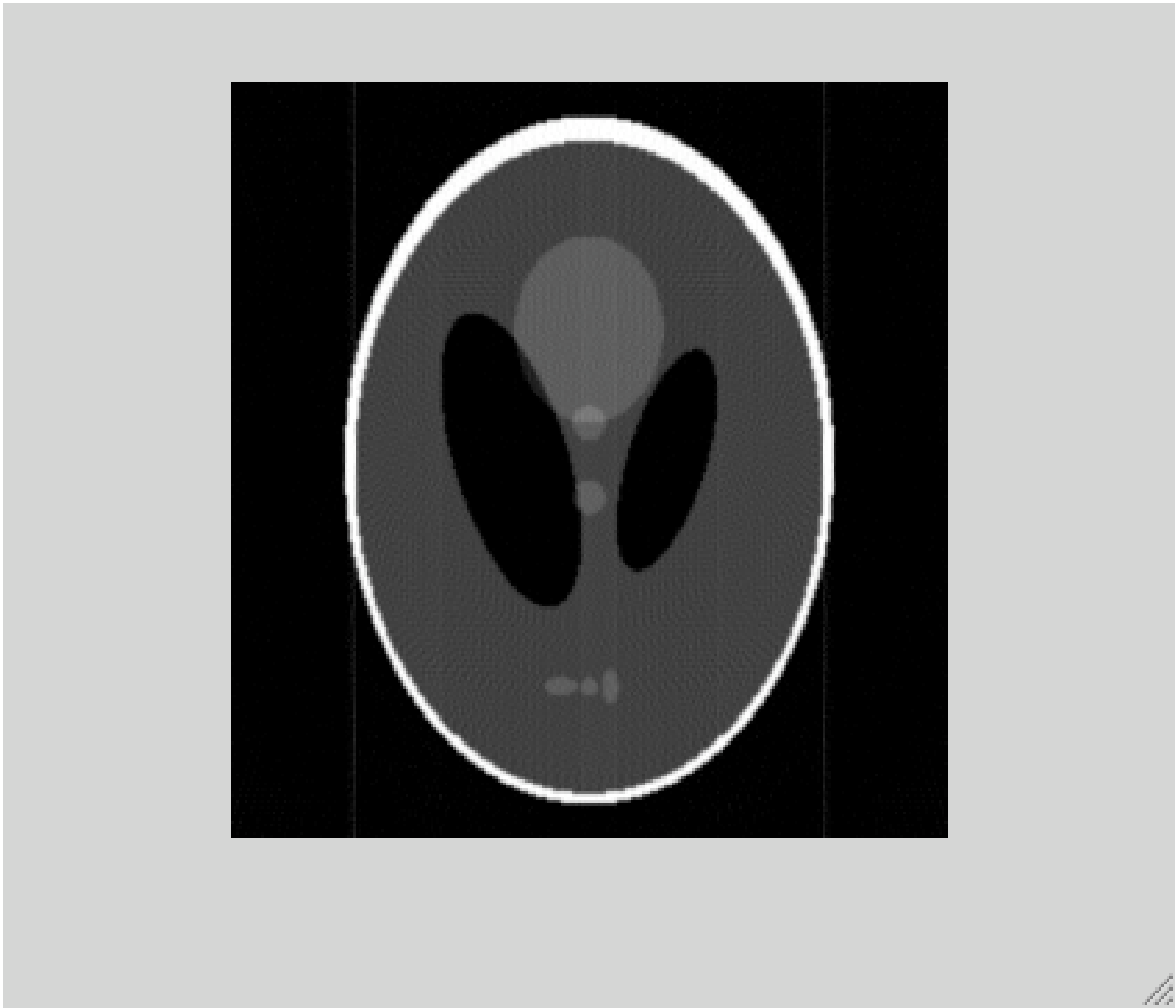
Reconstruction from fewer projection 0 to 45deg in steps of 1deg



0 to 100 deg in steps of 5 deg



0 to 180 in steps of 1 deg



0 to 180 in steps of 5 deg

Compressive Sampling/Sensing(CS):

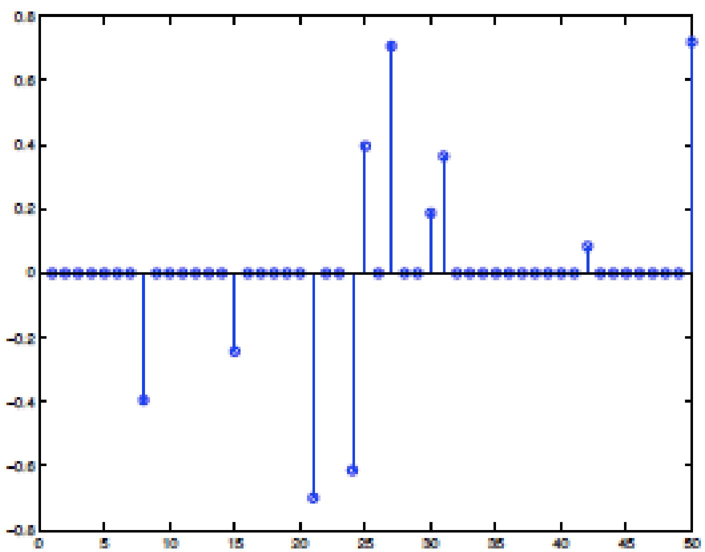
- It is possible to recover signal from fewer samples than that is dictated by the traditional sampling wisdom.
- it make use of the
    - Sparsity
        - natural signals are mostly sparse or can be compressed by using appropriate basis.
    - Incoherence :
        - sampling/sensing waveforms have dense representation in the sparsifying basis.

Sparsity and compressibility

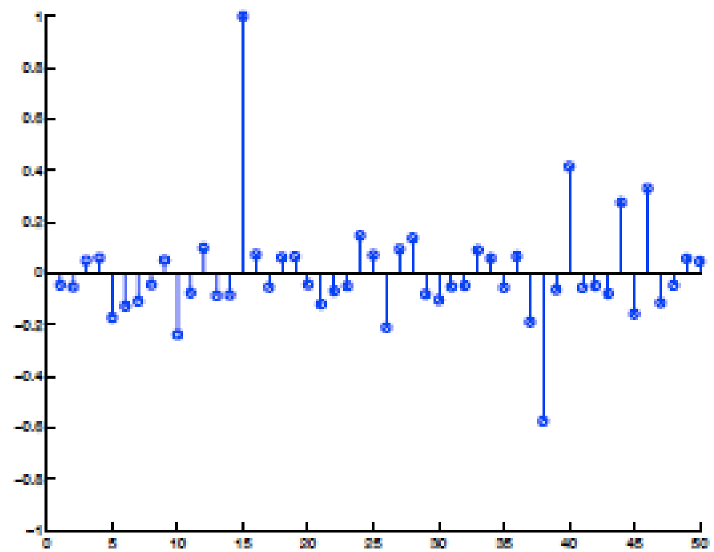
Expand image/signal f in an orthogonal basis Ψ

$$f(t) = \sum_{i=1}^{n} x_i \psi_i(t), \quad \text{or} \quad f = \Psi x$$

e.g. spikes, sinusoids, wavelets, curvelets, . . .



sparse



nearly sparse / compressible

Implication: can discard small coefficients without much perceptual loss

Wavelet approximation

Truncated wavelet expansion of $f = \sum_{i=1}^{n} x_i \psi_i$

$$f_S(t) = \sum_{S \text{ largest coeff's}} x_i \psi_i(t), \quad f_S = \Psi x_S$$

- Ψ matrix with columns $\psi_1, \psi_2, \ldots, \psi_n$
- x_S has S nonzero entries (S-sparse)
- nonzero entries of x_S are the S largest entries of x

1 megapixel image

25k term approximation

Incoherence

Signal expansion

$$f(t) = \sum_{j=1}^{n} x_j \psi_j(t)$$

- Sparsity (ortho)basis $\psi_j(t)$
- Measurement (ortho)basis $\phi_k(t)$

Definition (Coherence between Φ and Ψ)

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot \max_{1 \le k,j \le n} |\langle \phi_k, \psi_j \rangle|, \quad \|\psi_j\|_{\ell_2} = \|\phi_k\|_{\ell_2} = 1$$

Coherence obeys

$$1 \le \mu(\Phi, \Psi) \le \sqrt{n}$$

Example: spikes and sinusoids

- Fourier basis: $\psi_j(t) = n^{-1/2} e^{i2\pi jt/n}$
- Canonical or spike basis: $\phi_k(t) = \delta(t-k)$

Coherence is

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot \max_{k,j} |\langle \phi_k, \psi_j \rangle| = 1$$

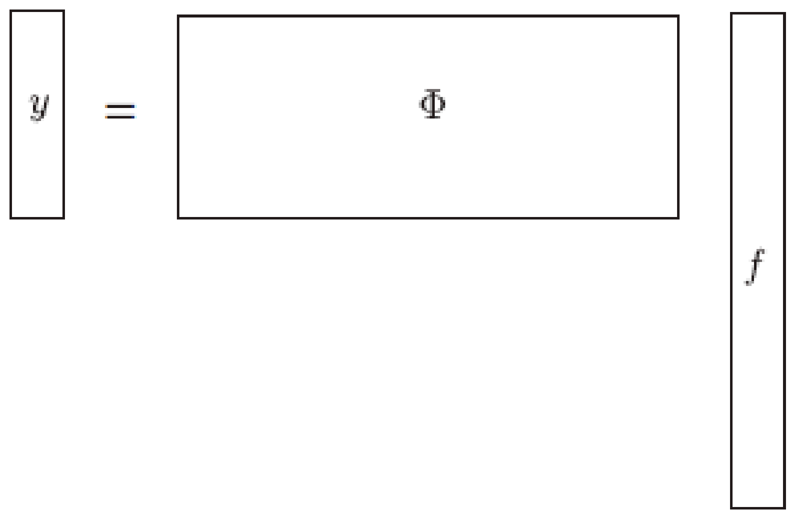
Spikes and sinusoids are maximally incoherent (true in any dimensions)

This is good because we want low coherence!

Signal recovery from undersampled data

Measure

$$y_k = \langle f, \phi_k \rangle, \quad k \in M, \quad |M| = m < n$$



Recover by ℓ_1 minimization ($\|x\|_{\ell_1} := \sum_i |x_i|$): $f^\star = \Psi x^\star$

$$x^\star = \text{argmin } \|\tilde{x}\|_{\ell_1} \quad \text{subject to} \quad y = \Phi\Psi\tilde{x}$$

Reformulation as a linear program (LP)

$$\min \sum_{i=1}^{n} |x_i| \quad \text{s.t.} \quad y = Ax$$

$$\Leftrightarrow$$

$$\min \sum_{i=1}^{n} u_i \quad \text{s.t.} \quad \begin{aligned} & y = Ax \\ & -u_i \leq x_i \leq u_i, \quad i = 1, \ldots, n \end{aligned}$$

with variables $u, x \in \mathbb{R}^n$

Very easy to solve!

Example

- Take $m = 96,000$ incoherent measurements $y = \Phi f_S$
- f_S = wavelet approximation (perfectly sparse)
- Solve (Ψ: wavelet transform)

$$\min \|x\|_{\ell_1} \quad \text{subject to} \quad \Phi\Psi x = y$$

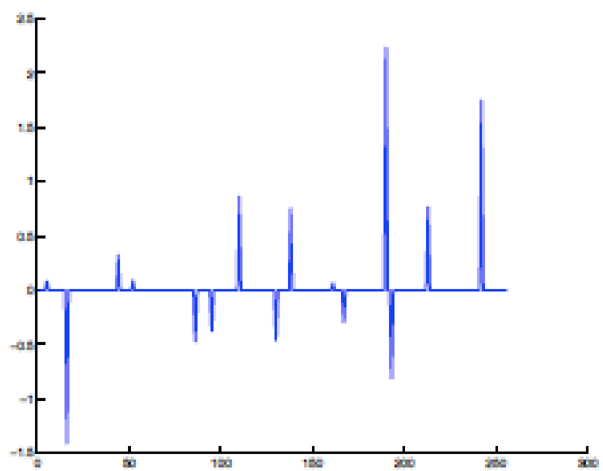
original (25k wavelets)

perfect recovery

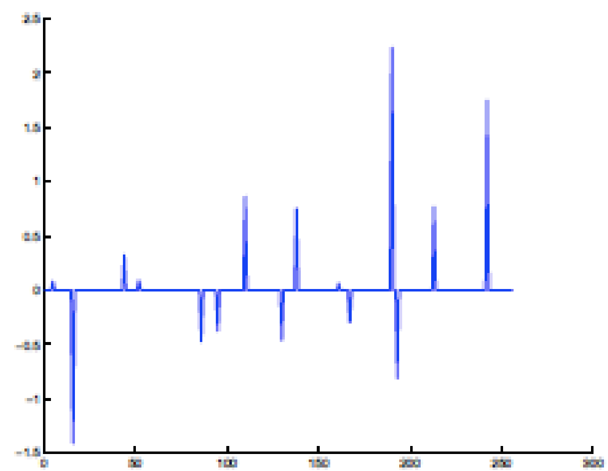
ℓ_1 reconstruction from undersampled freq. data

- signal length: $n = 256$
- data: 30 (complex) valued Fourier coefficients
- sparsity: $S = 15$

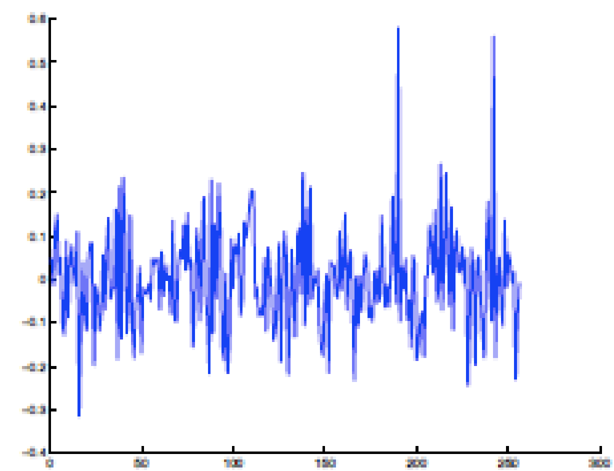
$$\min \|\tilde{x}\|_{\ell_1} \quad \text{s.t.} \quad \Phi\tilde{x} = y$$



original



ℓ_1 recovery = perfect!

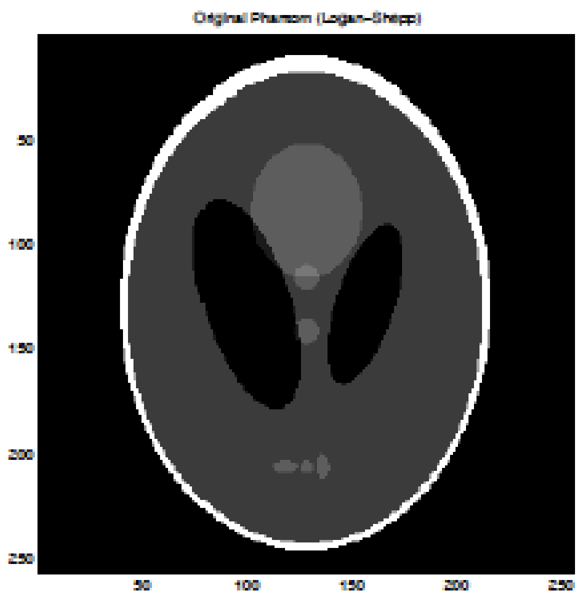


ℓ_2 recovery

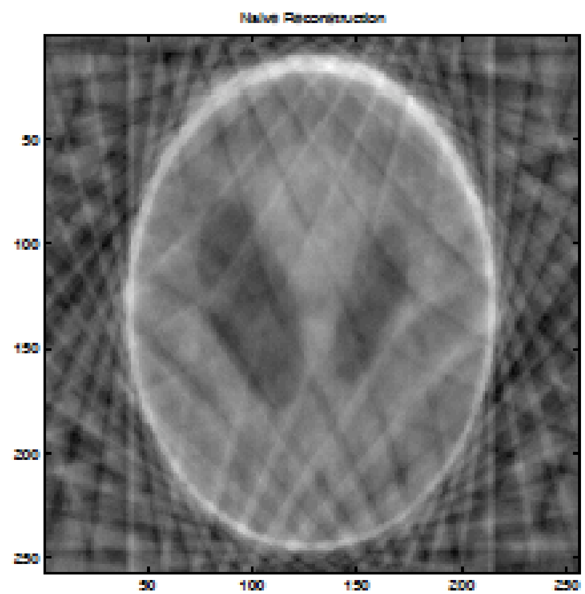
ℓ_1 reconstruction from undersampled freq. data

Reconstruct f with

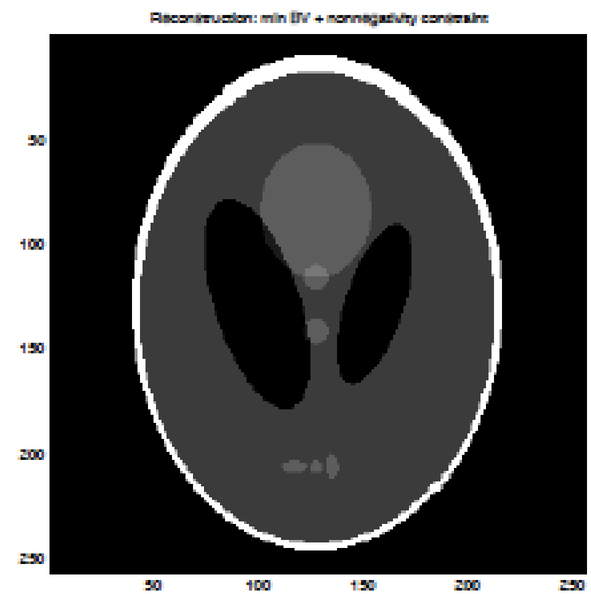
$$\min_s \|s\|_{TV} := \sum_{t_1,t_2} |\nabla s(t_1,t_2)| \quad \text{s.t.} \quad \hat{s}(\omega) = \hat{f}(\omega), \;\; \omega \in \Omega$$



original



filtered backprojection



perfect recovery

In Conclusion:

- to have a successful recovery/reconstruction of signal/images from under sampled data.
  - make sure the signal/image is sparse or generally transform using a sparsifying basis.
  - choose measuring waveforms in such a way that they will be incoherent with the sparsifing basis.
  - to recover the signal perform l1 optimization and linear programing.

Potential Applications:

- Magnetic Resonance Imaging(MRI)
- Geophysics

Thank you

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Questions?