MATH 5343 Homework 1 Spring 2011

Due: 3:00 p.m. on Wednesday, March 2, 2011

Exercises from: *Finite Difference Methods for Ordinary and Partial Differential Equations* by R. J. LeVeque, SIAM, 2007.

Exercise 1 *(derivation of finite difference formula)*

Recall the finite difference approximation to $u'(x)$

$$D_h^2 u(x) = \frac{1}{2h}[3u(x) - 4u(x-h) + u(x-2h)].$$

We derived this formula using the method of undetermined coefficients in class. Derive the same formula using the polynomial $p(x)$ that interpolates u at $x, x-h, x-2h$.

Exercise 2 *(use of* `fdstencil`*)*

(a) Use the method of undetermined coefficients to set up the 5×5 linear algebraic system that would determine a fourth-order accurate finite difference approximation to $u''(x)$ based on 5 equally spaced points,

$$u''(x) = c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0 u(x) + c_1 u(x+h) + c_2 u(x+2h) + O(h^4).$$

(b) Compute the coefficients using the MATLAB code `fdstencil.m` available from the website, and check that they satisfy the system you determined in part (a).

(c) Test this finite difference formula to approximate $u''(1)$ for $u(x) = \sin(2x)$ with values of h from the array `hvals = logspace(-1, -4, 13)`. Make a table of the error vs. h for several values of h and compare against the predicted error from the leading term of the expression printed by `fdstencil`. You may want to look at the m-file `example1.m` for guidance on how to make such a table.

Also produce a log-log plot of the absolute value of the error vs. h.

You should observe the predicted accuracy for larger values of h. For smaller values, numerical cancellation in computing the linear combination of u values impacts the accuracy observed.

Exercise 3 *(boundary conditions in* `bvp` *codes)*

(a) Modify the m-file `bvp2.m` so that it implements a Dirichlet boundary condition at $x = a$ and a Neumann condition at $x = b$ and test the modified program.

(b) Make the same modification to the m-file `bvp4.m`, which implements a fourth order accurate method. Again test the modified program.

Exercise 4 *(accuracy on nonuniform grids)*

Consider the 3-point approximation to $u''(x_i)$ based on $u(x_{i-1})$, $u(x_i)$, and $u(x_{i+1})$ using general nonuniform grid points x_{i-1}, x_i, and x_{i+1}. It is determined that the truncation error of this approximation is $\frac{1}{3}(h_{i-1} - h_i)u'''(x_i) + O(h^2)$, where $h_{i-1} = x_i - x_{i-1}$ and $h_i = x_{i+1} - x_i$, so the approximation is only first order accurate in h if h_{i-1} and h_i are $O(h)$ but $h_{i-1} \neq h_i$.

The program `bvp2.m` is based on using this approximation at each grid point. Hence on a nonuniform grid the local truncation error is $O(h)$ at each point, where h is some measure of the grid spacing (e.g., the average spacing on the grid). If we assume the method is stable, then we expect the global error to be $O(h)$ as well as we refine the grid.

(a) However, if you run `bvp2.m` you should observe second-order accuracy, at least provided you take a smoothly varying grid (e.g., set `gridchoice = 'rtlayer'` in `bvp2.m`). Verify this.

(b) What average order of accuracy is observed on a random grid? To test this, set `gridchoice = 'random'` in `bvp2.m` and increase the number of tests done, e.g., by setting `mvals = round(logspace(1,3,50));` to do 50 tests for values of m between 10 and 1000.

Exercise 5 *(ill-posed boundary value problem)*

Consider the following linear boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} &u''(x) + u(x) = 0 \quad \text{for } a < x < b \\ &u(a) = \alpha, \quad u(b) = \beta. \end{aligned}$$

Note that this equation arises from a linearized pendulum, for example.

(a) Modify the m-file `bvp2.m` to solve this problem. Test your modified routine on the problem with

$$a = 0, \quad b = 1, \quad \alpha = 2, \quad \beta = 3.$$

Determine the exact solution for comparison.

(b) Let $a = 0$ and $b = \pi$. For what values of α and β does this boundary value problem have solutions? Sketch a family of solutions in a case where there are infinitely many solutions.

(c) Solve the problem with

$$a = 0, \quad b = \pi, \quad \alpha = 1, \quad \beta = -1.$$

using your modified `bvp2.m`. Which solution to the boundary value problem does this appear to converge to as $h \to 0$? Change the boundary value at $b = \pi$ to $\beta = 1$. Now how does the numerical solution behave as $h \to 0$?

(d) You might expect the linear system in part (c) to be singular since the boundary value problem is not well posed. It is not, because of discretization error. Compute the eigenvalues of the matrix A for this problem and show that an eigenvalue approaches 0 as $h \to 0$. Also show that $\|A^{-1}\|_2$ blows up as $h \to 0$ so that the discretization is unstable.