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On a Weak Discrete Maximum Principle for *hp*-FEM

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Abstract: *In this paper we formulate and prove a new discrete maximum principle (DMP) for the Poisson equation in one spatial dimension solved by means of hp finite elements of degrees at most ten. While the DMP for piecewise-linear elements is a classical result from the 1970s, no extensions to the hp-FEM have been known, except for a negative result from 1981. We explain why it is not possible to make a straightforward extension of the classical DMP to higher-order elements, and propose stronger assumptions on the right-hand side under which an extension is possible.*

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1 Introduction

Discrete maximum principles (DMP) are the numerical counterparts of the (continuous) maximum principles for elliptic and parabolic PDEs. In the 1970s, these results were used to prove the convergence of finite differences and lowest-order finite element methods (see, e.g., [3, 4]). Nowadays the DMP still play an important role in computational PDEs by providing restrictions on the mesh and data under which the approximation of physically nonnegative quantities such as the density, temperature, concentration, or electric charge remains nonnegative. In early 1980s, Höhn and Mittelmann [7] proved that a straightforward generalization of a DMP to quadratic Lagrange elements did not hold but under unrealistic restrictions on the triangulation, and since then no new results on DMP for higher-order elements have been obtained. Also the current research on DMP deals exclusively with lowest-order elements (see, e.g., [8, 9, 10, 17, 18]).

In the last decades, significant progress has been made in the development of the *hp*-FEM and its applications to challenging large-scale problems in computational science and engineering (see, e.g., [1, 2, 5, 11, 12, 13, 15]). An increasing demand for these methods naturally implies a need for the generalization of the DMP from lowest-order to higher-order elements.

The outline of the present paper is as follows: The *hp*-FEM discretization of one-dimensional Poisson equation is briefly recalled in Section 2. An alternative proof of the classical DMP for piecewise-linear FEM is given in Section 3. In Section 4 we show why a straightforward extension of the standard DMP to the higher-order case is not possible. In Section 5 we formulate a reasonably weakened version of the standard DMP and prove it for finite elements of the polynomial degrees $p = 2, 3, \dots, 10$.

2 Model Problem and Its Discretization

Consider an open bounded interval $\Omega = (a, b) \subset \mathbb{R}$ and the Poisson equation $-u'' = f \quad \text{in}\Omega$ equipped with the homogeneous Dirichlet boundary conditions, $u(a) = u(b) = 0$. The standard variational formulation of this problem reads: Given a right-hand side $f \in L^2(\Omega)$, find a function $u \in H^1_0(\Omega)$ such that the identity

$$\int_a^b u'(x)v'(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x \tag{1}$$

holds for all test functions $v \in V$, where $V = H^1_0(\Omega) = W^{1,2}_0(\Omega)$ is the standard Sobolev space. We can restrict ourselves to the homogenous Dirichlet boundary conditions since the case of nonhomogeneous Dirichlet boundary conditions does not cause any difficulty nor does it involve special considerations,

Consider a grid $a = x_0 < x_1 < x_2 \ldots < x_M = b$ that subdivides $\overline{\Omega}$ into $M \geq 1$ finite elements $K_1, K_2, \ldots, K_M$. Each element K_i is equipped with a polynomial degree $1 \leq p_i \leq 10$ (the polynomial degrees can vary). The elements $K_1, K_2, \ldots, K_M$, equipped with the polynomial degrees $p_1, p_2, \ldots, p_M$, form a finite element mesh T_{hp}. The finite element space $V_{hp} \subset V$ over the mesh T_{hp} has the form

$$V_{hp} = \{v \in V;\ v(a) = v(b) = 0;\ v|_{K_i} \in P^{p_i}(K_i), 1 \leq i \leq M\}. \qquad (2)$$

Here the symbol $P^{p_i}(K_i)$ stands for the space of polynomials of degree less than or equal to p_i in the interval K_i. The dimension of this space is $\dim(V_{hp}) = -1 + \sum_{i=1}^{M} p_i$.

The finite element problem reads: Find a function $u_{hp} \in V_{hp}$ such that the identity

$$\int_a^b u'_{hp}(x) v'_{hp}(x)\,\mathrm{d}x = \int_a^b f(x) v_{hp}(x)\,\mathrm{d}x \qquad (3)$$

holds for every test function $v_{hp} \in V_{hp}$. Obviously, there exist unique solutions to both the continuous problem (1) and the discrete problem (3) (see, e.g., [12]).

3 Classical DMP for Piecewise-Linear FEM

At the beginning let us mention the piecewise-linear case ($p_1 = p_2 = \ldots = p_M = 1$). The classical DMP for the discrete problem (3) can be stated in several equivalent ways, from which we choose the following:

Lemma 3.1. *Let $p_1 = p_2 = \ldots = p_M = 1$, and let the right-hand side f of discrete problem (3) be nonnegative a.e. in Ω. Then the solution u_{hp} attains its minimum on the boundary $\partial\Omega$.*

Proof. The standard proof of this assertion is based on the theory of M-matrices (for details see the fundamental book [16] or the more recent publication [6]). Let us briefly mention an alternative proof for reference:

Consider a pair of adjacent elements $K_j = [x_{j-1}, x_j]$ and $K_{j+1} = [x_j, x_{j+1}]$, and the standard piecewise-linear "hat function" $v_j \in V_{hp}$ associated with the grid point x_j, such that $v_j(x_k) = \delta_{jk}$, $0 \leq k \leq M$, $1 \leq j \leq M - 1$, as shown in Figure 1.

Substituting now v_j for v_{hp} into the discrete problem (3) and using the nonnegativity of both f and v_j, we obtain

$$\int_{x_{j-1}}^{x_{j+1}} u'_{hp}(x) v'_j(x)\,\mathrm{d}x = \int_{x_{j-1}}^{x_{j+1}} f(x) v_j(x)\,\mathrm{d}x \geq 0. \qquad (4)$$

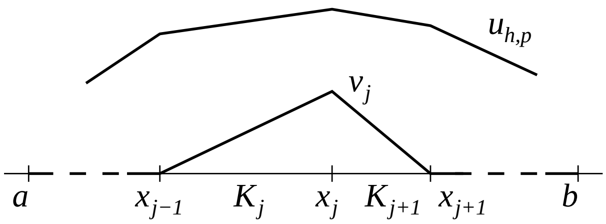

Figure 1: Classical DMP in the piecewise-linear case.

By $Du_{hp}^{(j)}$ and $Du_{hp}^{(j+1)}$ let us denote the constant slopes of the piecewise-linear function u_{hp} in the elements K_j and K_{j+1}, respectively. Using the fact that the slopes of the test function v_j in the elements K_j and K_{j+1} are $1/(x_j - x_{j-1})$ and $-1/(x_{j+1} - x_j)$, respectively, from the inequality (4) we immediately obtain

$$0 \le Du_{hp}^{(j)} \frac{x_j - x_{j-1}}{x_j - x_{j-1}} - Du_{hp}^{(j+1)} \frac{x_{j+1} - x_j}{x_{j+1} - x_j} = Du_{hp}^{(j)} - Du_{hp}^{(j+1)}.$$

Therefore, $Du_{hp}^{(j+1)} \le Du_{hp}^{(j)}$ for every internal grid point x_j, $1 \le j \le M-1$. Thus, the function u_{hp} is concave in Ω. Taking into account its zero values at Ω-endpoints, we conclude that u_{hp} attains its minimum on the boundary of Ω. □

4 Attempt of Straightforward Extension to *hp*-FEM

In this section we explain why a straightforward extension of Lemma 3.1 generally does not hold for higher-order elements. Let us begin with recalling the Lobatto shape functions [12, 13] for reference. These functions possess H_0^1-orthogonality which plays an important role in the proof of the weakened DMP in Section 5.

The Lobatto shape functions are defined in the interval $[-1, 1]$ by

$$l_k(x) \quad = \quad \frac{1}{\|L_{k-1}\|_{L^2}} \int_{-1}^{x} L_{k-1}(\xi)\,\mathrm{d}\xi, \quad 2 \le k, \tag{5}$$

where $L_0, L_1, \ldots$ are the Legendre polynomials with $\|L_{k-1}\|_{L^2} = \sqrt{2/(2k-1)}$. It follows from (5) that the functions $l_2, l_3, \ldots$ vanish at both $x = -1$ and $x = 1$, and that they are orthonormal in the $H_0^1(-1, 1)$ product,

$$(l_i, l_j)_{H_0^1(-1,1)} = \int_{-1}^{1} l_i'(x) l_j'(x)\,\mathrm{d}x = \delta_{ij}, \quad 2 \le i, j. \tag{6}$$

It follows from (6) that the Lobatto shape functions are optimal for the discretization of the Laplace operator. Explicit formulae of the functions $l_2, l_3, \ldots, l_{10}$ are mentioned in the Appendix for reference.

Example ($f \geq 0$ and u_{hp} attains its minimum in the interior of Ω)

Let $\Omega = (a, b) = (-1, 1)$,

$$f(x) = 200\mathrm{e}^{-10(x+1)}, \tag{7}$$

and consider a finite element mesh T_{hp} consisting of a single cubic element $K_1 = [a, b]$. The basis of the corresponding finite element space V_{hp} comprises the quadratic and cubic Lobatto shape functions l_2 and l_3, and the solution u_{hp} has the form

$$u_{hp}(x) = y_1 l_2(x) + y_2 l_3(x).$$

By (6) the stiffness matrix is the identity matrix, and the unknown coefficients y_1, y_2 have the form

$$y_i = \int_{-1}^{1} \sum_{j=1}^{2} y_j l'_{j+1}(x) l'_{i+1}(x)\,\mathrm{d}x = \int_{-1}^{1} f(x) l_{i+1}(x)\,\mathrm{d}x, \quad i = 1, 2. \tag{8}$$

Using the right-hand side (7), we obtain

$$\begin{aligned} y_1 &= -\sqrt{6}\left(9 + 11\mathrm{e}^{-20}\right)/10, \\ y_2 &= \sqrt{10}\left(73 - 133\mathrm{e}^{-20}\right)/100. \end{aligned}$$

Thus the solution u_{hp} has the form

$$u_{hp}(x) = \frac{1}{40}\left(1 - x^2\right)\left(54 + 66\mathrm{e}^{-20} - (73 - 133\mathrm{e}^{-20})x\right). \tag{9}$$

Since u_{hp} is not nonnegative in Ω (see Figure 2), the attempt for a straightforward extension of the standard DMP failed.

To understand what happened, let us introduce the L^2-projection $f_{hp} \in V_{hp}$ of the right-hand side $f \in L^2(\Omega)$ to the space V_{hp},

$$\int_a^b (f_{hp}(x) - f(x)) v_{hp}(x)\,\mathrm{d}x = 0 \quad \text{for all } v_{hp} \in V_{hp}. \tag{10}$$

(When expressing f_{hp} as a linear combination of basis functions of V_{hp} and substituting into (10), one obtains a system of linear algebraic equations for the corresponding unknown coefficients.) The L^2-projection of the right-hand side (7),

$$f_{hp}(x) = \frac{3}{80}\left(1 - x^2\right)\left(110\mathrm{e}^{-20} + 90 + (931\mathrm{e}^{-20} - 511)x\right) \tag{11}$$

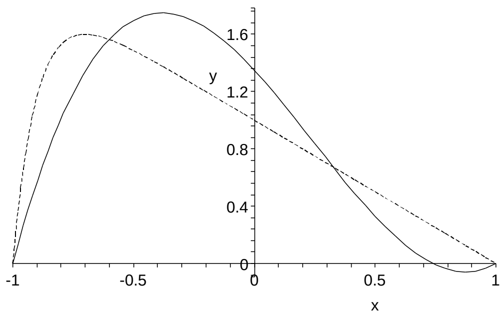

Figure 2: The hp-FEM solution u_{hp} with minimum in the interior of $(-1, 1)$ (solid line) and the exact solution u with minimum at the endpoints (dashed line).

is negative in a subset of Ω, as illustrated in Figure 3. Notice that (10) implies

$$\int_a^b f_{hp}(x)v_{hp}(x)\,\mathrm{d}x = \int_a^b f(x)v_{hp}(x)\,\mathrm{d}x \quad \text{for all } v_{hp} \in V_{hp}.$$

Therefore it does not matter whether f or f_{hp} stands on the right-hand side of the discrete problem (3). In other words, the hp-FEM solution (9) depicted in Figure 2 corresponds to the right-hand side (11) which is not nonnegative in Ω.

5 Weak Discrete Maximum Principle for *hp*-FEM

The previous example motivates us to formulate the following theorem:

Theorem 5.1. *Let $\Omega = (a, b) \subset \mathbb{R}$. Consider the discrete problem (3) on a mesh T_{hp} consisting of M finite elements $K_1, K_2, \ldots, K_M$ of polynomial degrees $1 \le p_1, p_2, \ldots, p_M \le 10$. Let the right-hand side $f \in L^2(\Omega)$ be such that its L^2-projection to V_{hp}, defined by* (10), *is nonnegative in* Ω. *Then the hp-FEM solution u_{hp} attains its minimum on the boundary of* Ω.

Proof: First we may exclude the trivial case $f_{hp}(x) \equiv 0$. By a standard result for the Laplace operator in one dimension, it is $u_{hp}(x_i) = u(x_i)$ for all $i = 0, 1, \ldots, M$, where u is the exact solution. Taking into account the (continuous) maximum principle, we have $u \ge 0$ in Ω and thus $u_{hp}(x_i) \ge 0$ for all $i = 0, 1, \ldots, M$. Therefore it is sufficient to prove Theorem 5.1 for a single element $K_1 = \Omega = (-1, 1)$ and polynomial degrees $p = 2, 3, \ldots, 10$.

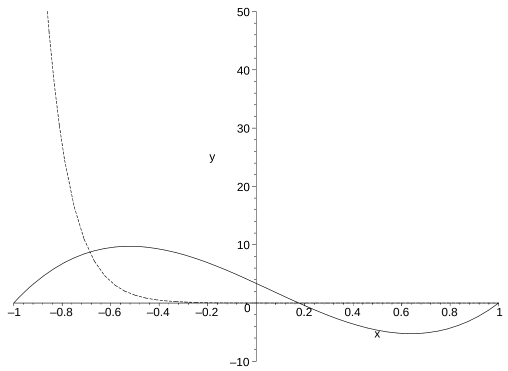

Figure 3: The right-hand side (7) (dashed line) and its L^2-projection (11) to the space V_{hp} (solid line).

Let us begin with the quadratic case:

Quadratic case ($p = 2$):

The solution to the discrete problem (3) is sought in the form

$$u_{hp}(x) = y_1 l_2(x). \quad (12)$$

By (8) and (10),

$$y_1 = \int_{-1}^{1} f_{hp}(z) l_2(z) \, \mathrm{d}z. \quad (13)$$

Since $l_2 < 0$ in Ω, $f_{hp} \geq 0$ in Ω, and $f_{hp} \not\equiv 0$ in Ω, necessarily $y_1 < 0$ and therefore, $u_{hp} > 0$ in Ω.

Cubic case ($p = 3$):

The cubic and higher-order cases are more involved since the higher-order Lobatto shape functions $l_3, l_4, \ldots$ change sign in Ω. The solution u_{hp} is sought in the form

$$u_{hp}(x) = y_1 l_2(x) + y_2 l_3(x), \quad (14)$$

where by (8)

$$y_1 = \int_{-1}^{1} f_{hp}(z) l_2(z) \, \mathrm{d}z, \quad y_2 = \int_{-1}^{1} f_{hp}(z) l_3(z) \, \mathrm{d}z. \quad (15)$$

It is easy to check that in the cubic case the conditions

$$u'_{hp}(-1) \ \geq \ 0, \quad (16)$$
$$u'_{hp}(1) \ \leq \ 0, \quad (17)$$

together with the homogeneous Dirichlet boundary conditions $u_{hp}(\pm 1) = 0$, are sufficient to guarantee that the minimum of u_{hp} in Ω is attained on the boundary. Using (14), (15), and the formulae for the Lobatto shape functions l_2 and l_3, condition (16) translates into

$$0 \le u'_{hp}(-1) = y_1 l'_2(-1) + y_2 l'_3(-1)$$
$$= -\sqrt{3/2}\int_{-1}^{1} f_{hp}(z) l_2(z)\,\mathrm{d}z + \sqrt{5/2}\int_{-1}^{1} f_{hp}(z) l_3(z)\,\mathrm{d}z$$
$$= \frac{1}{4}\int_{-1}^{1} f_{hp}(z) \underbrace{(z^2-1)(5z-3)}_{=g_a(z)}\,\mathrm{d}z = \frac{1}{4}\int_{-1}^{1} f_{hp}(z) g_a(z)\,\mathrm{d}z.$$

Analogously, condition (17) can be reformulated to

$$0 \le \frac{1}{4}\int_{-1}^{1} f_{hp}(z) \underbrace{(z^2-1)(-5z-3)}_{=g_b(z)}\,\mathrm{d}z = \frac{1}{4}\int_{-1}^{1} f_{hp}(z) g_b(z)\,\mathrm{d}z.$$

It is sufficient to show that for every nonnegative cubic polynomial f_{hp}, both L^2-products $(f_{hp}, g_a)_{L^2}$ and $(f_{hp}, g_b)_{L^2}$ are nonnegative. For symmetry reasons (see Figure 4) we restrict ourselves to the case $(f_{hp}, g_a)_{L^2}$ only.

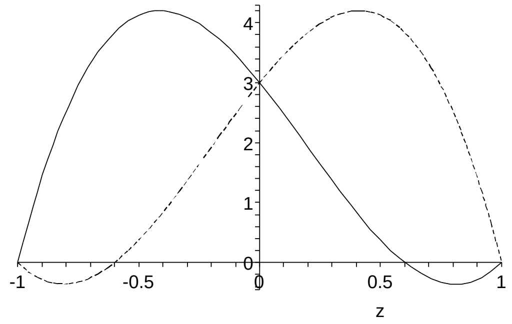

Figure 4: The function $g_a(z)$ (solid line) and $g_b(z)$ (dashed line).

To cover all forms that a nonnegative cubic function f_{hp} can have, one has to distinguish between the following eighteen cases:

1. $f_{hp} = c$, where $c > 0$ is a constant,
2. f_{hp} is a nonconstant affine function with
   (a) positive slope and a root in the interval $(-\infty, -1]$,

(b) negative slope and a root in the interval $[1, \infty)$,

3. f_{hp} is a quadratic function with

   (a) two complex-conjugate complex roots and a positive leading term,

   (b) one real root of multiplicity two and a positive leading term,

   (c) two roots in $(-\infty, -1]$ and a positive leading term,

   (d) two roots in $[1, \infty)$ and a positive leading term,

   (e) one root in $(-\infty, -1]$, one root in $[1, \infty)$, and a negative leading term,

4. f_{hp} is a cubic function with a positive leading term and

   (a) one single root in $(-\infty, -1]$ and one root of multiplicity two in $\mathbb{R}$,

   (b) one root in $(-\infty, -1]$ and two real roots in $[1, \infty)$,

   (c) one root in $(-\infty, -1]$ and two complex-conjugate complex roots,

   (d) three different roots in $(-\infty, -1]$,

   (e) one root of multiplicity three in $(-\infty, -1]$,

5. f_{hp} is a cubic function with a negative leading term and conditions (a)–(e) analogous to the previous case.

Using a tedious calculation, we proved that the L^2-product $(f_{hp}, g_a)_{L^2}$ is nonnegative in all these cases. Let us illustrate the procedure at least in the cases 4(a) and 4(b):

Case 4(a): It is $f_{hp}(z) = (z-c)^2(z+d)$, where $c \in \mathbb{R}$ and $d \geq 1$. Therefore,

$$\int_{-1}^{1} f_{hp}(z) g_a(z)\, \mathrm{d}z = d \underbrace{\left(4c^2 + \frac{8}{3}c + \frac{4}{5}\right)}_{\geq 0 \text{ for all } c \in \mathbb{R}} - \frac{4}{7} - \frac{8}{5}c - \frac{4}{3}c^2$$

$$\geq \left(4c^2 + \frac{8}{3}c + \frac{4}{5}\right) - \frac{4}{7} - \frac{8}{5}c - \frac{4}{3}c^2 = \frac{8}{3}c^2 + \frac{16}{15}c + \frac{8}{35} \geq 0.$$

Case 4(b): It is $f_{hp}(z) = (z-c)(z-d)(z+e)$, where $c, d, e \geq 1$ are such that (without loss of generality) $d = c + \varepsilon$ with $\varepsilon > 0$. We obtain

$$\int_{-1}^{1} f_{hp}(z) g_a(z)\, \mathrm{d}z = -\frac{4}{7} - \frac{4}{5}c + \frac{4}{5}e + \frac{4}{3}ce - \frac{4}{5}d - \frac{4}{3}cd + \frac{4}{3}de + 4cde$$

$$= \left(\underbrace{\frac{4}{3}e - \frac{4}{5}}_{\geq 0} + \underbrace{\left(4e - \frac{4}{3}\right)}_{\geq 0} c\right)\varepsilon + \underbrace{\left(4e - \frac{4}{3}\right)}_{\geq 0} c^2 + \underbrace{\left(\frac{8}{3}e - \frac{8}{5}\right)}_{\geq 0} c + \underbrace{\frac{4}{5}e - \frac{4}{7}}_{\geq 0} \geq 0$$

However, an extension of the previous analysis to higher-order elements is not obvious since (a) conditions analogous to (16) and (17) are difficult to find and (b) the corresponding discrete inequalities become very complicated. Because of this, let us introduce an alternative way of proving the quartic and higher-order cases ($p \geq 4$):

On the element $K_1 = (-1, 1)$ equipped with a general polynomial degree $p \geq 4$, the solution u_{hp} is sought in the form

$$u_{hp}(x) = \sum_{i=1}^{p-1} y_i l_{i+1}(x), \tag{18}$$

analogously to (14) and (15). By (6) the relation (8) can be extended to

$$y_i = \int_{-1}^{1} f_{hp}(z) l_{i+1}(z) \, \mathrm{d}z. \tag{19}$$

Putting (19) into (18), we obtain

$$u_{hp}(x) = \sum_{i=1}^{p-1} \left(\int_{-1}^{1} f_{hp}(z) l_{i+1}(z) \, \mathrm{d}z \right) l_{i+1}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_p(x, z) \, \mathrm{d}z, \tag{20}$$

where

$$\Phi_p(x, z) = \sum_{i=1}^{p-1} l_{i+1}(x) l_{i+1}(z).$$

Using the formulae for the Lobatto shape functions (23), it is easy to write down the polynomial kernel $\Phi_p(x, z)$ for any degree $2 \leq p \leq 10$ explicitly. Since $l_{i+1}(\pm 1) = 0$ for all $i = 1, 2, \ldots$, it is

$$\Phi_p(x, z) = 0 \quad \text{for all } (x, z) \in \Gamma, \tag{21}$$

where $\Gamma = \overline{(-1, 1)^2} \setminus (-1, 1)^2$. The kernel Φ_p, which is nothing else than the discrete Green's function of the problem, will play an important role in the proof for all $p \geq 4$.

Quartic case ($p = 4$):

The function $\Phi_4(x, z)$ is a fourth-degree polynomial in both x and z. Its gradient is a vector-valued cubic function. It can be verified analytically that

$$\nabla \Phi_4(x_0, z_0) = (0, 0)^T \text{ in } (-1, 1)^2$$

if and only if $(x_0, z_0) = (0, 0)$. Since $\Phi_4(0, 0) > 0$, it is $\Phi_4 > 0$ in $(-1, 1)^2$ and by (20) we have

$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_4(x, z) \, \mathrm{d}z > 0 \quad \text{for all } x \in \Omega. \tag{22}$$

Using the assumptions $f_{hp} \geq 0$ in Ω and $f_{hp} \not\equiv 0$ in Ω, it follows that the minimum of u_{hp} is attained on the boundary of Ω. For illustration, the function Φ_4 is depicted in Figure 5. (The shapes of Φ_5, Φ_6, ... are similar.)

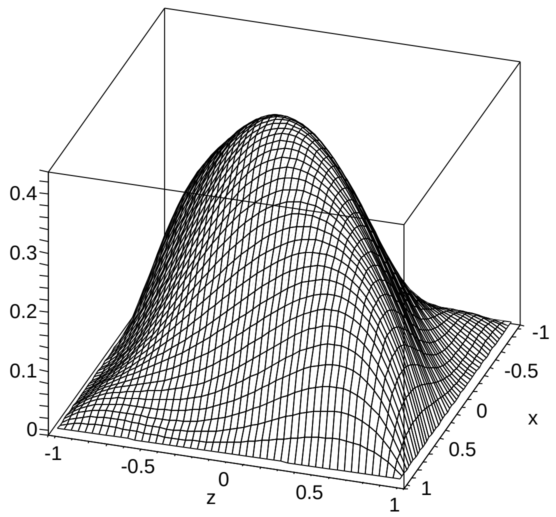

Figure 5: The discrete Green's function Φ_4.

Quintic case ($p = 5$):

Now, unfortunately, the function $\Phi_5(x, z)$ is negative in the corners of $(-1, 1)^2$, as shown in Figure 6.

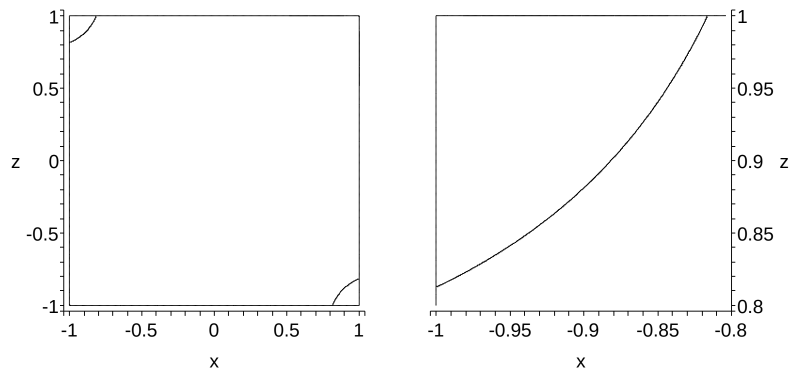

Figure 6: Zero level set of the kernel $\Phi_5(x, z)$ (left) and a detail of the upper left corner (generated by Maple).

It can be verified analytically that Φ_5 only is negative within the squares $\Omega_1 = (-1, -0.811) \times (0.811, 1)$ and $\Omega_2 = (0.811, 1) \times (-1, -0.811)$. Recall the relation (20) for $p = 5$,

$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z)\Phi_5(x,z)\,\mathrm{d}z.$$

Since $f_{hp}(z) \in P^5(\Omega)$, we have that $f_{hp}(z)\Phi_5(x,z) = F_x^{(10)}(z) \in P^{10}(\Omega)$. For symmetry, we can restrict ourselves to the case $x \in [0,1)$.

Clearly, $F_x^{(10)}(z) \geq 0$ for any $x \in [0,1)$ and $z \in [-0.811, 1)$. Moreover, by (21) it is $F_x^{(10)}(-1) = 0$. Hence, the proof for $p = 5$ can be concluded if we find a quadrature rule $\mathcal{Q}_{10}$ with the following properties:

1. $\mathcal{Q}_{10}$ is exact for all 10th-degree polynomials in Ω.
2. $\mathcal{Q}_{10}$ uses positive weights $w_0, w_1, \ldots, w_{10}$ only.
3. $\mathcal{Q}_{10}$ uses points $z_0, z_1, \ldots, z_{10}$ in $[-1,1]$ but outside of $(-1, -0.811)$.

Then all the values $F_x^{(10)}(z_i)$ are nonnegative. Moreover, since $f_{hp} \not\equiv 0$ in Ω, at least one of these values is positive. For any $x \in [0,1)$ we obtain

$$u_{hp}(x) = \int_{-1}^{1} F_x^{(10)}(z)\,\mathrm{d}z = \sum_{i=0}^{10} \underbrace{w_i}_{>0} \underbrace{F_x^{(10)}(z_i)}_{\geq 0} > 0.$$

An example of a quadrature rule $\mathcal{Q}_{10}$ meeting the above properties 1.–3. is given in Table 1.

Table 1: Case $p = 5$; 10th-order quadrature rule in Ω with positive weights and points lying in $[-1,1]$ but outside of $(-1, -0.811)$.

Point	Weight	Point	Weight
-1	0.0534286192	-0.811	0.3054087580
-0.59	0.0030544353	-0.42	0.4473230113
-0.2	0.0066984041	0	0.2760767276
0.2	0.2939694773	0.43	0.0149245373
0.6	0.3805105712	0.9	0.1999066353
1	0.0186988234		

Remark 5.1. *Speculations on limited accuracy of the quadrature rule can be avoided by realizing that (a) the quadrature points shown in Table 1 are exact, and (b) the weights were truncated for printing purposes only. In reality, for*

each point $z_0, z_1, \dots, z_{10}$ the exact value of the weight w_i is obtained via the formula

$$w_i = \int_{-1}^{1} \mathcal{L}_i(x)\, dx,$$

where $\mathcal{L}_i \in P^{10}(-1,1)$ is the elementary Lagrange interpolation polynomial, $\mathcal{L}_i(z_j) = \delta_{ij}$, $0 \le i,j \le 10$. The same applies to all quadrature rules in the following.

By the existence of this quadrature rule, we conclude that for $p = 5$ the minimum of u_{hp} in Ω always is attained on the boundary.

Case $p = 6$:

In this case it is $\Phi_6(x,z) > 0$ in the entire domain Ω^2. Therefore, as in the fourth-order case,

$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z)\Phi_6(x,z)\,\mathrm{d}z > 0 \quad \text{for all } x \in \Omega,$$

and we can conclude immediately that the minimum of u_{hp} in Ω always is attained on the boundary.

Case $p = 7$:

The behavior of the function $\Phi_7(z,x)$ is similar to $\Phi_5(x,z)$, as illustrated in Figure 5.

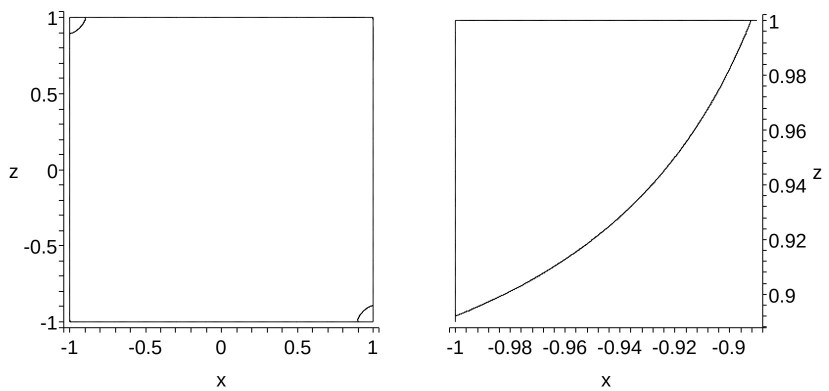

Figure 7: Zero level set of the kernel $\Phi_7(x,z)$ (left) and a detail of the upper left corner (generated by Maple).

This time the kernel $\Phi_7(x,z) > 0$ in $\Omega^2 \backslash (\Omega_1 \cup \Omega_2)$, where $\Omega_1 = (-1,-0.89) \times (0.89,1)$ and $\Omega_2 = (0.89,1) \times (-1,-0.89)$. It follows from the existence of a fourteenth-order quadrature rule $\mathcal{Q}_{14}$ with positive weights and points lying outside of $(-1,-0.89)$ that $u_{hp}(x) > 0$ for all $x \in [0,1)$. For illustration, a concrete example of a quadrature rule $\mathcal{Q}_{14}$ is shown in Table 2.

Table 2: Case $p = 7$; 14th-order quadrature rule in Ω with positive weights and points lying in $[-1,1]$ but outside of $(-1,-0.89)$.

Point	Weight	Point	Weight
-1	0.0306200311	-0.89	0.1806438688
-0.75	0.0016558668	-0.65	0.2862680475
-0.45	0.0379885258	-0.31	0.2988638595
-0.16	0.0833146476	0.1	0.3554921618
0.16	0.0113639321	0.35	0.0204292124
0.47	0.3218682171	0.734	0.1289561668
0.80	0.1314089188	0.955	0.1093567805
1	0.0017697634		

Again, by symmetry it also is $u_{hp}(x) > 0$ for all $x \in (-1,0]$, and we can conclude that the minimum of u_{hp} in Ω always is attained on the boundary.

Case $p = 8$:

There are four areas where the function $\Phi_8(x,z)$ is negative (see Figure 8). Two of them lie inside the rectangles $(-1,-0.98) \times (0.75, 0.85)$ and $(-0.85,-0.75) \times (0.98,1)$, and the other two are located symmetrically at the opposite corner of Ω^2.

In order to prove that $u_{hp}(x) > 0$ for $x \in (-1,-0.98]$, it is sufficient to find a quadrature rule $\mathcal{Q}_{16}$ with positive weights and points outside $(0.75, 0.85)$. To prove that $u_{hp}(x) > 0$ for $x \in (-0.98, 0]$ we need another quadrature rule $\mathcal{Q}'_{16}$ with positive weights and points outside of $(0.98, 1)$. The points and weights of such quadrature rules are listed in Tables 3 and 4. Also in this case the proof is concluded by symmetry argument.

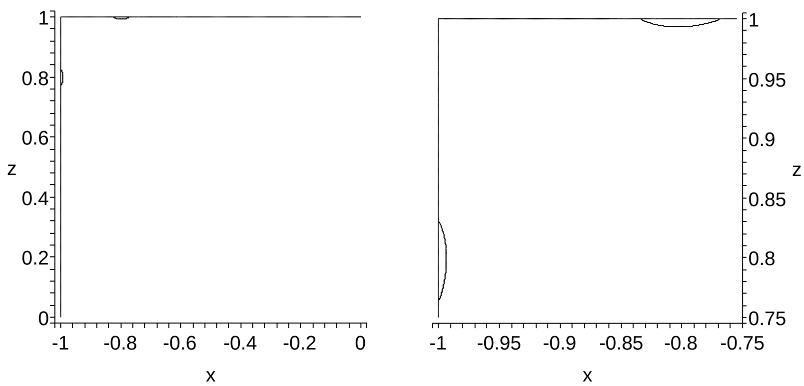

Figure 8: Zero level set of the kernel $\Phi_8(x, z)$ in the second quadrant (left) and a detail of the upper left corner.

Table 3: Case $p = 8$; 16th-order quadrature rule in Ω with positive weights and points lying in $[-1, 1]$ but outside of $(0.75, 0.85)$.

Point	Weight	Point	Weight
-1	0.0137599529	-0.9564181650	0.0618586932
-0.8854980347	0.0892150513	-0.7582972896	0.1646935265
-0.5719162652	0.1875234174	-0.4628139806	0.0729252387
-0.2917166274	0.2435469772	-0.0811621291	0.0841621866
-0.0061521460	0.1800939083	0.1655560030	0.1320371771
0.3391628868	0.2286184297	0.5726348225	0.2184036287
0.75	0.1285378345	0.85	0.0908051678
0.9230637084	0.0427456544	0.9648584341	0.0509010934
1	0.0101720626		

Case $p = 9$:

The function $\Phi_9(x, z)$ is similar to $\Phi_5(x, z)$ and $\Phi_7(x, z)$, as illustrated in Figure 9.

The kernel $\Phi_9(x, z)$ is positive in $\Omega^2 \setminus (\Omega_1 \cup \Omega_2)$, where $\Omega_1 = (-1, -0.93) \times (0.93, 1)$ and $\Omega_2 = (0.93, 1) \times (-1, -0.93)$. It follows from the existence of an eighteenth-order quadrature rule $\mathcal{Q}_{18}$ with positive weights and points outside of $(-1, -0.93)$, that $u_{hp}(x) > 0$ for all $x \in [0, 1)$. An example of such quadrature rule is shown in Table 5. By symmetry, it is $u_{hp}(x) > 0$ for all $x \in (-1, 0]$, which concludes the proof of this case.

Case $p = 10$:

Table 4: Case $p = 8$; 16th-order quadrature rule in Ω with positive weights and points lying in $[-1, 1]$ but outside of $(0.98, 1)$.

Point	Weight	Point	Weight
-1	0.0097495069	-0.9548248562	0.0857520162
-0.8409569422	0.1018591390	-0.7825414112	0.0149475627
-0.7708636219	0.0926211201	-0.5747624113	0.2476049720
-0.3937499257	0.0549434125	-0.3273530867	0.0276562411
-0.2532942335	0.2543287199	0.0382371812	0.2892622856
0.2837396038	0.1910189889	0.4501581170	0.1560300966
0.5808907063	0.1246581226	0.7443822112	0.1842879621
0.8927849373	0.0841645246	0.9421667341	0.0612885001
1	0.0198268291		

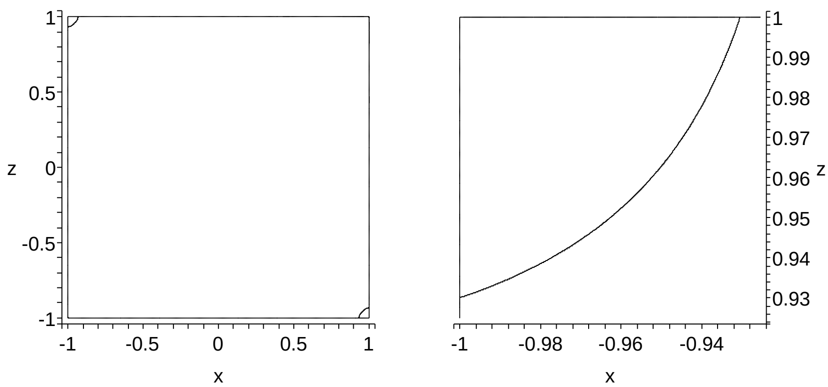

Figure 9: Zero level set of the kernel $\Phi_9(x, z)$ (left) and a detail of the upper left corner (generated by Maple).

There are four areas where the function $\Phi_{10}(x, z)$ is negative, analogously to the 8th-order case (see Figure 10). Two of them are inside the rectangles $(-1, -0.986) \times (0.82, 0.91)$ and $(-0.91, -0.82) \times (-0.986, 1)$ and the other two are located symmetrically at the opposite corner of Ω^2.

To prove that $u_{hp}(x) > 0$ for $x \in (-1, -0.986)$, it is sufficient to find a 20th-order quadrature rule $\mathcal{Q}_{20}$ with positive weights and points outside $(0.82, 0.91)$. To prove that $u_{hp}(x) > 0$ for $x \in (-0.986, 0]$, we need a 20th-order quadrature rule $\mathcal{Q}'_{20}$ with positive weights and points outside $(0.986, 1)$. The points and weights of such quadrature rules are listed in Tables 6 and 7, respectively.

By symmetry, $u_{hp} > 0$ for $x \in [0, 1)$ and thus the minimum of u_{hp} in Ω

Table 5: Case $p = 9$; 18th-order quadrature rule in Ω with positive weights and points lying in $[-1, 1]$ but outside of $(-1, -0.93)$.

Point	Weight	Point	Weight
-1	0.01937406240	-0.93	0.1153128270
-0.885	0.00157968340	-0.772	0.1947443595
-0.65	0.00126499680	-0.55	0.2341166464
-0.4	0.06286669339	-0.25	0.2438572426
-0.08	0.08588496537	0.08	0.2395820916
0.19	0.04691799156	0.38	0.2665159766
0.6	0.00216030838	0.625	0.2029738760
0.73	0.04687189997	0.83	0.1072052560
0.89	0.06009091818	0.97	0.0648680095
1	0.00381219535		

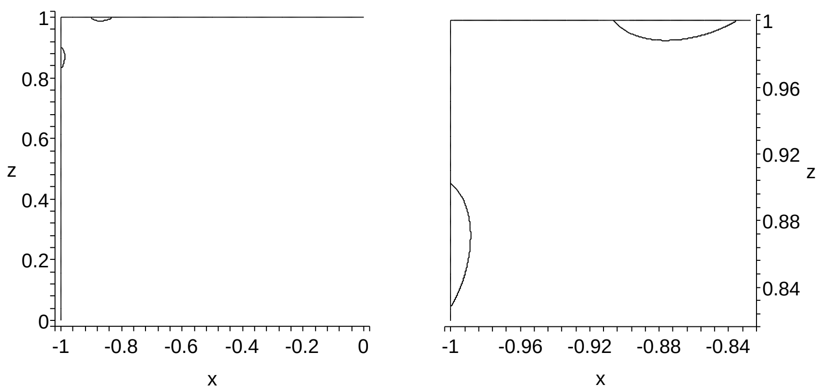

Figure 10: Zero level set of the kernel $\Phi_{10}(x, z)$ in the second quadrant (left) and a detail of the upper left corner.

always is attained on the boundary. Herewith, the proof of Theorem 5.1 is complete. $\square$

Remark 5.2. *Note that the discrete Green's function Φ_p was nonnegative in the entire $(-1, 1)^2$ for $p = 2$, $p = 4$, and $p = 6$. This means that on quadratic, quartic, and sextic elements even the original, strong version of the discrete maximum principle holds.*

Table 6: Case $p = 10$; 20th-order quadrature rule in Ω with positive weights and points lying in $[-1, 1]$ but outside of $(0.82, 0.91)$.

Point	Weight	Point	Weight
-1	0.0127411726	-0.9569019461	0.0603200758
-0.9344466123	0.0183508422	-0.8574545411	0.1032513172
-0.7530104489	0.1106942630	-0.6362178184	0.0412386636
-0.6061244531	0.1295220930	-0.4275824090	0.1937516842
-0.2340018112	0.1916905139	-0.0454114485	0.1774661870
0.0754465671	0.0755419308	0.1672504233	0.0745275871
0.2516247645	0.1488965177	0.3707975798	0.0207086237
0.4366736344	0.1397170181	0.5306011976	0.0924918512
0.6745457042	0.1639628301	0.82	0.1200387168
0.91	0.0649445615	0.9667274132	0.0502362251
1	0.0099073255		

Table 7: Case $p = 10$; 20th-order quadrature rule in Ω with positive weights and points lying in $[-1, 1]$ but outside of $(0.986, 1)$.

Point	Weight	Point	Weight
Points: -1	0.0129961117	-0.9609467424	0.0393058650
-0.9366001558	0.0472129994	-0.8686571459	0.0307704321
-0.8222969304	0.1127110155	-0.6830858117	0.1442049485
-0.5515874908	0.1263749495	-0.4070028385	0.1615584597
-0.2391731402	0.1767071143	-0.0805321378	0.0223802647
-0.0404112041	0.1755155830	0.0382998004	0.0409103698
0.2054285570	0.2302298514	0.4168373782	0.1495405342
0.4862170553	0.0877842194	0.6284448676	0.0980645550
0.6932595712	0.1047143177	0.83041757281	0.1311485592
0.93562906418	0.0774056021	0.986	0.0267375743
1	0.0037266735		

Conclusions

In this paper we used the one-dimensional Poisson equation to demonstrate that the assumption of nonnegativity of the right-hand side f is not sufficient to guarantee discrete maximum principles for the hp-FEM. We introduced a stronger assumption that the L^2 projection f_{hp} of the right-hand side f on the finite element space V_{hp} be nonnegative. The proof presented in this paper was done for polynomial degrees less than or equal to ten, which is sufficient for most practical hp-FEM computations. For any polynomial degree $p > 10$, the proof can be extended in a straightforward fashion by locating the zero level set of the discrete Green's function Φ_p and constructing a corresponding positive quadrature rule $\mathcal{Q}_{2p}$.

It remains to be explained that the software Maple was used to locate approximately the zero level sets of the discrete Green's functions Φ_p. After that, rigorous proof of their nonnegativity in $(-1,1)^2$ minus these areas was performed using an adaptive interval computation technique in integer arithmetics. More details on this step can be found in [14]. Generalization of this proof to higher spatial dimensions and more general elliptic operators is subject of our current research.

Appendix: Lobatto shape functions $l_2, l_3, \ldots, l_{10}$

For the sake of self-containedness we present the explicit formulae of the H^1_0-orthonormal shape functions (5) based on integrated Legendre polynomials (for more details see, e.g., [12, 13]):

$$
\begin{aligned}
l_2(x) &= \frac{1}{2}\sqrt{\frac{3}{2}}(x^2-1), \\
l_3(x) &= \frac{1}{2}\sqrt{\frac{5}{2}}(x^2-1)x, \\
l_4(x) &= \frac{1}{8}\sqrt{\frac{7}{2}}(x^2-1)(5x^2-1), \\
l_5(x) &= \frac{1}{8}\sqrt{\frac{9}{2}}(x^2-1)(7x^2-3)x, \\
l_6(x) &= \frac{1}{16}\sqrt{\frac{11}{2}}(x^2-1)(21x^4-14x^2+1), \\
l_7(x) &= \frac{1}{16}\sqrt{\frac{13}{2}}(x^2-1)(33x^4-30x^2+5)x, \\
l_8(x) &= \frac{1}{128}\sqrt{\frac{15}{2}}(x^2-1)(429x^6-495x^4+135x^2-5), \\
l_9(x) &= \frac{1}{128}\sqrt{\frac{17}{2}}(x^2-1)(715x^6-1001x^4+385x^2-35)x, \\
l_{10}(x) &= \frac{1}{256}\sqrt{\frac{19}{2}}(x^2-1)(2431x^8-4004x^6+2002x^4-308x^2+7).
\end{aligned}
\tag{23}
$$

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