

The University of Texas at El Paso  
Department of Mathematical Sciences  
Research Reports Series

El Paso, Texas

Research Report No. 2006-02

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*Discrete Maximum Principle  
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**Abstract:** *We formulate a sufficient condition on the mesh under which we prove the discrete maximum principle (DMP) for the one-dimensional Poisson equation with Dirichlet boundary conditions discretized by the  $hp$ -FEM. The DMP holds if a relative length of every element  $K$  in the mesh is bounded by a value  $H_{\text{rel}}^*(p) \in [0.9, 1]$ , where  $p \geq 1$  is the polynomial degree of the element  $K$ . The values  $H_{\text{rel}}^*(p)$  are calculated for  $1 \leq p \leq 100$ .*

**AMS subject classification:** 35B50, 65N60, 74S05

**Keywords:** Discrete maximum principle, discrete Green's function, higher-order elements,  $hp$ -FEM, Poisson equation

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### Acknowledgment

The first author has been supported by the Grant Agency of the Czech Republic, project No. 201/04/P021 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503. The second author has been supported in part by the U.S. Department of Defense under Grant No. 05PR07548-00, by the NSF Grant No. DMS-0532645, and by Grant Agency of the Czech Republic projects No. 102-05-0629. This support is gratefully acknowledged.

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# 1 Introduction

Classical (continuous) maximum principles belong to the most important results in the theory of second-order partial differential equations (PDEs). Their discrete counterparts, discrete maximum principles (DMP), appeared in the early 1970s. They were used by various authors to prove the convergence of the lowest-order finite difference and finite element methods (see, e.g., [2, 3] and the references therein). DMP have been studied intensively during the past decades in the context of linear PDEs [1, 7, 9, 16, 17, 18] and more recently also nonlinear equations [8]. Most of these results have two points in common:

- They are limited to lowest-order approximations.
- They are based on M-matrices [5, 15].

Much less is known about the DMP for methods of higher orders of accuracy such as higher-order finite difference methods, spectral FEM, or  $hp$ -FEM. Let us mention, e.g., a result [19] on higher-order collocation methods. Particularly noteworthy is a negative result [6] from 1981 stating that a stronger DMP is not valid for cubic and higher-order Lagrange elements in 2D. In the quadratic case, the stronger DMP is valid under extremely restrictive assumptions on the mesh, which almost never could be satisfied in practice. In the light of this negative result, a few attempts were made to formulate and prove weakened forms of the DMP (see, e.g., [10, 13]). The present result is based on the analysis of the discrete Green's function (DGF) for higher-order elements. Similar concept was used in the piecewise-linear case in [4].

The paper is organized as follows: In Section 2 we introduce the one-dimensional Poisson problem, its  $hp$ -FEM discretization, and the discrete maximum principle. The discrete Green's function along with its basic properties is discussed in Section 3. In Section 4 we derive an explicit formula for the DGF for the Poisson problem discretized by  $hp$ -FEM, which is used to find sufficient conditions for its nonnegativity in Section 5. This leads to the notion of critical relative element length  $H_{\text{rel}}^*$ . The main result is presented in Section 6.

## 2 Model Problem and Its Discretization

We consider the one-dimensional Poisson equation with homogeneous Dirichlet boundary conditions in an open bounded interval  $\Omega = (a, b)$ . The standard weak formulation reads: Find  $u \in V = H_0^1(\Omega)$  such that

$$a(u, v) = (f, v) \quad \forall v \in V, \tag{1}$$

where  $f \in L^2(\Omega)$ , the symbol  $(\cdot, \cdot)$  stands for the inner product in  $L^2(\Omega)$ ,  $H_0^1(\Omega)$  is the standard Sobolev space, and  $a(u, v) = (u', v')$ .

We create a partition  $a = x_0 < x_1 < \dots < x_M = b$  of the domain  $\Omega$  consisting of  $M$  elements  $K_i = [x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, M$ . Every element  $K_i$  is assigned an arbitrary polynomial degree  $p_i \geq 1$ . The corresponding finite element space of piecewise-polynomial continuous functions  $V_{hp} \subset V$  has the form

$$V_{hp} = \{v_{hp} \in V; v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \dots, M\},$$

where  $P^{p_i}(K_i)$  stands for the space of polynomials of degree at most  $p_i$  on the element  $K_i$ . The space  $V_{hp}$  has the dimension  $N = -1 + \sum_{i=1}^M p_i$ . There exists a unique function  $u_{hp} \in V_{hp}$  satisfying

$$a(u_{hp}, v_{hp}) = (f, v_{hp}) \quad \forall v_{hp} \in V_{hp}. \quad (2)$$

**Definition 2.1.** *We say that problem (2) satisfies the discrete maximum principle (DMP) if for any right-hand side  $f \in L^2(\Omega)$  it holds*

$$f \geq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } \Omega.$$

**Remark 2.1.** *The above implication is equivalent to*

$$f \geq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad \min_{x \in \bar{\Omega}} u_{hp}(x) = \min_{x \in \partial\Omega} u_{hp}(x)$$

*for homogeneous Dirichlet boundary conditions. This is further equivalent to*

$$f \leq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad \max_{x \in \bar{\Omega}} u_{hp}(x) = \max_{x \in \partial\Omega} u_{hp}(x).$$

**Remark 2.2.** *In problem (2), homogeneous Dirichlet conditions are considered without loss of generality. This follows immediately from the fact that every solution  $\hat{u}_{hp}$  to a problem with nonhomogeneous Dirichlet boundary conditions can be written as  $\hat{u}_{hp} = u_{hp}^L + u_{hp}$ , where  $u_{hp}^L$  is a linear function satisfying the nonhomogeneous conditions, and  $u_{hp}$  vanishes at  $\Omega$ -endpoints.*

### 3 Discrete Green's Function

The discrete Green's function (DGF) is defined in analogy with the standard (continuous) Green's function:

**Definition 3.1.** *For an arbitrary  $z \in \Omega$ , the unique solution  $G_{hp,z} \in V_{hp}$  to the problem*

$$a(v_{hp}, G_{hp,z}) = v_{hp}(z) \quad \forall v_{hp} \in V_{hp} \quad (3)$$

*is called the discrete Green's function (DGF) corresponding to the point  $z$ .*

In the following, we will use the notation  $G_{hp}(x, z) = G_{hp,z}(x)$ . A combination of (2) and (3) yields an important consequence

$$u_{hp}(z) = \int_{\Omega} G_{hp}(x, z) f(x) dx \quad \forall z \in \Omega. \quad (4)$$

The following lemma shows that the DGF can easily be expressed using any basis of  $V_{hp}$ , cf. [4]. We use the Kronecker symbol

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases}$$

**Lemma 3.1.** *Let  $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$  be any basis of  $V_{hp}$ . If the stiffness matrix  $A_{ij} = a(\varphi_j, \varphi_i)$ ,  $1 \leq i, j \leq N$ , is nonsingular, then*

$$G_{hp}(x, z) = \sum_{j=1}^N \sum_{k=1}^N A_{jk}^{-1} \varphi_k(x) \varphi_j(z). \quad (5)$$

Here  $A_{jk}^{-1}$  are the entries of the inverse stiffness matrix, i.e.,  $\sum_{j=1}^N A_{ij} A_{jk}^{-1} = \delta_{ik}$ ,  $1 \leq i, k \leq N$ .

*Proof.* Substitute

$$G_{hp}(x, z) = \sum_{i=1}^N c_i(z) \varphi_i(x) \quad (6)$$

into (3) with  $v_{hp} = \varphi_j$ . It follows that

$$\sum_{i=1}^N c_i(z) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(z).$$

The coefficients  $c_i(z)$  can be expressed in terms of the inverse matrix as  $c_k(z) = \sum_{j=1}^N \varphi_j(z) A_{jk}^{-1}$ , and they can be substituted back into (6).  $\square$

**Corollary 3.1.** *Let  $\{l_1, l_2, \dots, l_N\}$  be a basis of  $V_{hp}$  such that  $a(l_i, l_j) = \delta_{ij}$ . Then*

$$G_{hp}(x, z) = \sum_{i=1}^N l_i(x) l_i(z).$$

**Lemma 3.2.** *If there exists a basis  $\{l_1, l_2, \dots, l_N\}$  of  $V_{hp}$  such that  $a(l_i, l_j) = \delta_{ij}$ ,  $1 \leq i, j \leq N$ , then  $G_{hp}(x, x) > 0$  for all  $x \in \Omega$ .*

*Proof.* Let  $x \in \Omega$ . Since  $\{l_1, l_2, \dots, l_N\}$  is a basis, there exists at least one  $k \in \{1, 2, \dots, N\}$  such that  $l_k(x) \neq 0$ . Hence, by Corollary 3.1

$$G_{hp}(x, x) = \sum_{i=1}^N l_i^2(x) > 0.$$

□

**Theorem 3.1.** *Problem (2) satisfies the discrete maximum principle if and only if the corresponding discrete Green's function  $G_{hp}(x, z) = G_{hp,z}(x)$  defined by (3) is nonnegative in  $\Omega^2$ .*

*Proof.* Immediate consequence of (4). □

**Remark 3.1.** *Results presented in this section are valid for any second-order elliptic problem of the form (1), also in higher spatial dimensions.*

## 4 DGF for Poisson Problem in 1D

### 4.1 Lowest-Order Case

Consider the case  $p_1 = p_2 = \dots = p_M = 1$  first. Let  $\mathcal{B}^L = \{\phi_1, \phi_2, \dots, \phi_{M-1}\}$  be the standard lowest-order basis consisting of the piecewise-linear ‘‘hat functions’’ such that  $\phi_j(x_i) = \delta_{ij}$ ,  $1 \leq i, j \leq M-1$ . In this case the stiffness matrix  $A^L \in \mathbb{R}^{(M-1) \times (M-1)}$  is tridiagonal,

$$A_{ij}^L = \begin{cases} 1/h_i + 1/h_{i+1} & \text{for } i = j, \\ -1/h_{i+1} & \text{for } i = j - 1, \\ -1/h_{i-1} & \text{for } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

with  $h_i = x_i - x_{i-1}$ .

**Proposition 4.1.** *The inverse matrix  $(A^L)^{-1} \in \mathbb{R}^{(M-1) \times (M-1)}$  has the form*

$$(A^L)^{-1} = \frac{1}{b-a} \begin{pmatrix} (x_1 - a)(b - x_1) & (x_1 - a)(b - x_2) & (x_1 - a)(b - x_3) & \dots \\ (x_1 - a)(b - x_2) & (x_2 - a)(b - x_2) & (x_2 - a)(b - x_3) & \dots \\ (x_1 - a)(b - x_3) & (x_2 - a)(b - x_3) & (x_3 - a)(b - x_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

*i.e.*,  $(A^L)_{ij}^{-1} = (x_i - a)(b - x_j)/(b - a)$  for  $1 \leq i \leq j \leq M-1$  and  $(A^L)_{ij}^{-1} = (x_j - a)(b - x_i)/(b - a)$  for  $1 \leq j < i \leq M-1$ .

*Proof.* This follows from a lengthy but straightforward computation of the product  $(A^L)^{-1}A^L$ . □

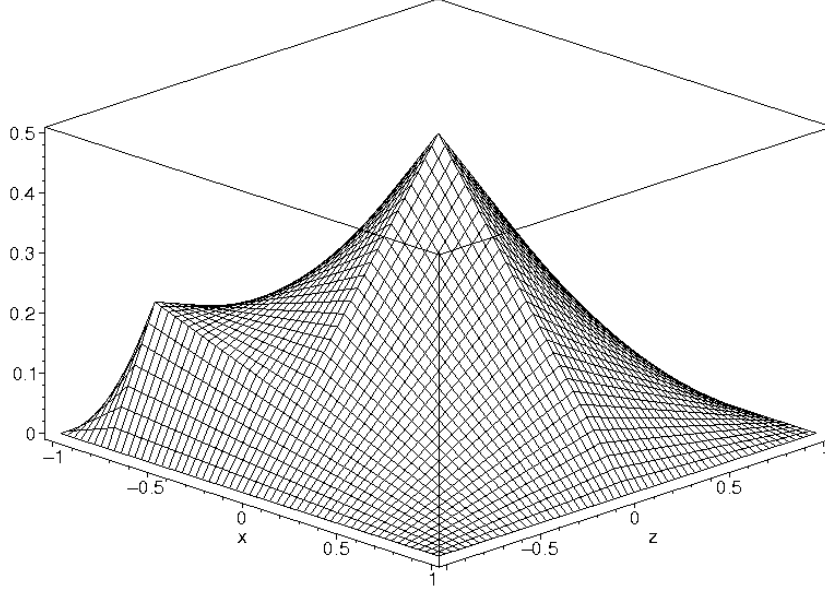


Figure 1: The lowest-order part  $G_{hp}^L(x, z)$  of the discrete Green's function  $G_{hp}(x, z)$  for the Poisson equation in  $\Omega = (-1, 1)$ , on a mesh with three elements  $[-1, -3/4]$ ,  $[-3/4, 0]$ , and  $[0, 1]$ .

Using Proposition 4.1 and identity (5), we can write the DGF in the form

$$G_{hp}^L(x, z) = \frac{1}{b-a} \left( \sum_{i=1}^{M-1} (x_i - a)(b - x_i) \phi_i(x) \phi_i(z) + \sum_{i=1}^{M-2} \sum_{j=i+1}^{M-1} (x_i - a)(b - x_j) [\phi_i(x) \phi_j(z) + \phi_j(x) \phi_i(z)] \right). \quad (7)$$

In particular, we see immediately that

$$G_{hp}^L(x, z) \geq 0 \quad \forall [x, z] \in \Omega^2. \quad (8)$$

The situation is illustrated in Figure 1.

## 4.2 Higher-Order Case

In this paragraph we return to the original setting with arbitrary polynomial degrees  $p_i \geq 1$ . In order to facilitate the construction of higher-order ba-

sis functions of the space  $V_{hp}$ , let us introduce the Lobatto shape functions  $l_0, l_1, l_2, \dots$  on a reference interval  $\hat{K} = [-1, 1]$  (see, e.g., [11, 14]).

The lowest-order Lobatto shape functions  $l_0$  and  $l_1$  have the form  $l_0(\xi) = (1-\xi)/2$ ,  $l_1(\xi) = (1+\xi)/2$ ,  $\xi \in \hat{K}$ . The higher-order shape functions  $l_2, l_3, \dots$  are defined as antiderivatives to the Legendre polynomials. Therefore, they satisfy

$$\int_{-1}^1 l'_i(\xi) l'_j(\xi) d\xi = \delta_{ij}, \quad i, j = 2, 3, \dots$$

Every Lobatto shape function  $l_i$ ,  $i = 2, 3, \dots$ , is a polynomial of degree  $i$  and it vanishes at  $\pm 1$ . Thus it can be expressed as

$$l_i(\xi) = l_0(\xi) l_1(\xi) \kappa_i(\xi), \quad i = 2, 3, \dots,$$

where  $\kappa_i$  is a polynomial of degree  $i - 2$ . For reference, a first few kernels  $\kappa_i$  are listed in Appendix.

The basis  $\mathcal{B} = \{\phi_1, \phi_2, \dots, \phi_N\}$  of  $V_{hp}$  can be written as  $\mathcal{B} = \mathcal{B}^L \cup \mathcal{B}^B$ , where  $\mathcal{B}^L$  was defined above and  $\mathcal{B}^B$  is the higher-order part of the basis comprising functions  $\phi_M, \phi_{M+1}, \dots, \phi_N$ . These are defined as follows:

Consider the standard linear transformations from  $\hat{K}$  to  $K_i$ ,

$$\chi_{K_i}(\xi) = \frac{(x_i - x_{i-1})\xi + (x_i + x_{i-1})}{2}. \quad (9)$$

On an element  $K_i$  of the polynomial degree  $p_i$ , there are  $p_i - 1$  higher-order basis functions. These vanish outside of  $K_i$  and in  $K_i$  they are defined as the Lobatto shape functions  $l_2, l_3, \dots, l_{p_i}$  composed with the inverse map  $\chi_{K_i}^{-1}(x)$ .

**Proposition 4.2.** *We have the following orthogonality relations:*

$$\begin{aligned} a(\phi^L, \phi^B) &= 0 \quad \forall \phi^L \in \mathcal{B}^L, \quad \forall \phi^B \in \mathcal{B}^B, \\ a(\phi^B, \psi^B) &= 0 \quad \forall \phi^B \in \mathcal{B}^B, \quad \forall \psi^B \in \mathcal{B}^B, \quad \phi^B \neq \psi^B. \end{aligned}$$

*Proof.* The proof is straightforward, based on the  $L^2$ -orthogonality of the Legendre polynomials.  $\square$

By Proposition 4.2, both the stiffness matrix  $A$  and its inverse have the following block structure:

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

with

$$D = \text{diag} \left( \underbrace{\frac{2}{h_1}, \dots, \frac{2}{h_1}}_{(p_1-1) \text{ times}}, \underbrace{\frac{2}{h_2}, \dots, \frac{2}{h_2}}_{(p_2-1) \text{ times}}, \dots, \underbrace{\frac{2}{h_M}, \dots, \frac{2}{h_M}}_{(p_M-1) \text{ times}} \right). \quad (10)$$

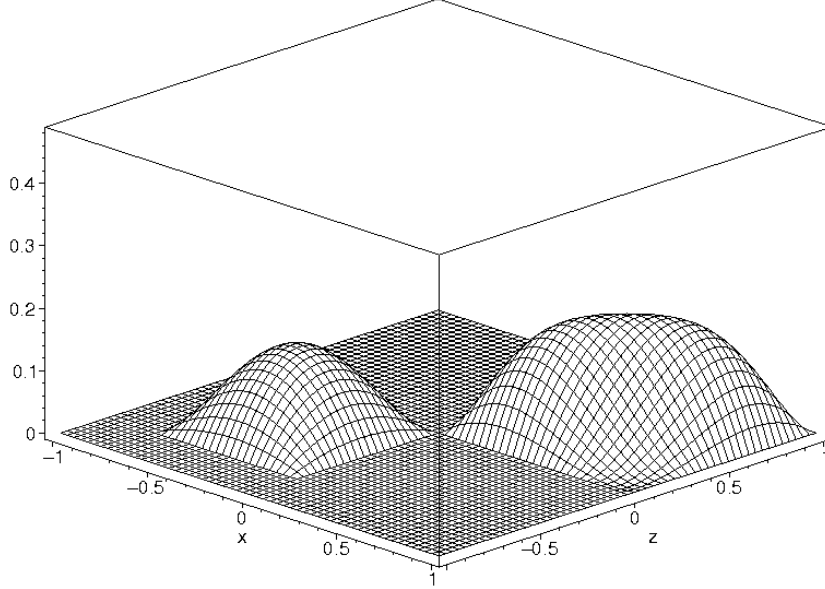


Figure 2: The higher-order part  $G_{hp}^B(x, z)$  of the discrete Green's function  $G_{hp}(x, z)$  for the Poisson equation in  $\Omega = (-1, 1)$ , on a mesh with three elements  $[-1, -3/4]$ ,  $[-3/4, 0]$ , and  $[0, 1]$  of the polynomial degrees  $p_1 = 1$ ,  $p_2 = 2$ ,  $p_3 = 3$ .

By (5), the DGF can be written as

$$G_{hp}(x, z) = G_{hp}^L(x, z) + G_{hp}^B(x, z), \quad (11)$$

where  $G_{hp}^L(x, z)$  corresponds to (7) and

$$G_{hp}^B(x, z) = \sum_{k=M}^N D_{kk}^{-1} \phi_k(x) \phi_k(z) \quad \forall [x, z] \in \Omega^2. \quad (12)$$

Unfortunately,  $G_{hp}^B(x, z)$  defined by (12) is not nonnegative in the entire  $\Omega^2$  in general. For instance, in the example shown in Figure 2, there are small regions near the points  $[1, 0]$  and  $[0, 1]$ , where the function  $G_{hp}^B(x, z)$  is negative.

Notice that any partition of  $\Omega$  produces a rectangular grid on  $\Omega^2$ , and that  $G_{hp}^B(x, z)$  can be nonzero within the diagonal squares of this grid only. In other words,

$$\text{supp } G_{hp}^B \subset \bigcup_{i=1}^M K_i^2. \quad (13)$$

**Lemma 4.1.** *The discrete Green's function  $G_{hp}$  defined by (11) is nonnegative in  $\Omega^2 \setminus \bigcup_{i=1}^M K_i^2$ .*

*Proof.* Consider (13) together with (8). □

## 5 The DGF on $K_i^2$

As justified by Lemma 4.1, we only need to continue with the study of the discrete Green's function  $G_{hp}(x, z)$  in the union of the diagonal squares  $\bigcup_{i=1}^M K_i^2$ . Without loss of generality, let us restrict ourselves to only one square  $K_i^2$ ,  $1 \leq i \leq M$ . Let  $p = p_i$  be the polynomial degree assigned to  $K_i$ . Notice that only a few terms in (7) and (12) are nonzero in  $K_i^2$ . Hence, by (7), (10), and (12) we obtain

$$\begin{aligned} G_{hp}(x, z)|_{K_i^2} &= \frac{(x_i - a)(b - x_i)}{b - a} \phi_i(x) \phi_i(z) \\ &\quad + \frac{(x_{i-1} - a)(b - x_{i-1})}{b - a} \phi_{i-1}(x) \phi_{i-1}(z) \\ &\quad + \frac{(x_{i-1} - a)(b - x_i)}{b - a} [\phi_i(x) \phi_{i-1}(z) + \phi_{i-1}(x) \phi_i(z)] \\ &\quad + \frac{x_i - x_{i-1}}{2} G_{hp}^B(x, z), \end{aligned} \quad (14)$$

$[x, z] \in K_i^2$ ,  $1 \leq i \leq M$ . It is convenient to introduce the notation  $K_i = [x_{i-1}, x_i] = [L, R]$ .

We transform the function  $G_{hp}$  from  $K_i^2$  to the reference square  $\hat{K}^2 = [-1, 1]^2$  using the linear transformation (9) with  $x = \chi_{K_i}(\xi)$  and  $z = \chi_{K_i}(\eta)$ ,

$$\begin{aligned} G_{hp}(x, z)|_{K_i^2} &= \hat{G}_{hp}(\xi, \eta) = \frac{(R - a)(b - R)}{b - a} l_1(\xi) l_1(\eta) \\ &\quad + \frac{(L - a)(b - L)}{b - a} l_0(\xi) l_0(\eta) \\ &\quad + \frac{(L - a)(b - R)}{b - a} [l_1(\xi) l_0(\eta) + l_0(\xi) l_1(\eta)] \\ &\quad + \frac{R - L}{2} \hat{G}_{hp}^{p,B}(\xi, \eta), \end{aligned} \quad (15)$$

$[\xi, \eta] \in \hat{K}^2$ . Here  $l_0(\xi)$  and  $l_1(\xi)$  are the above-defined lowest-order shape functions on  $\hat{K}$  and

$$\hat{G}_{hp}^{p,B}(\xi, \eta) = \sum_{k=2}^p l_k(\xi) l_k(\eta) = l_0(\xi) l_0(\eta) l_1(\xi) l_1(\eta) \sum_{k=2}^p \kappa_k(\xi) \kappa_k(\eta) \quad (16)$$

is the higher-order part.

Let us modify formula (15) in the following way: Divide (15) by  $R-L > 0$  and use the identities

$$\begin{aligned}\frac{(L-a)(b-L)}{(b-a)(R-L)} &= \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a}, \\ \frac{(R-a)(b-R)}{(b-a)(R-L)} &= \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{b-R}{b-a},\end{aligned}$$

and

$$l_0(\xi)l_0(\eta) + l_1(\xi)l_1(\eta) + l_0(\xi)l_1(\eta) + l_1(\xi)l_0(\eta) = 1 \quad \forall[\xi, \eta] \in \hat{K}^2.$$

We obtain

$$\begin{aligned}\frac{\hat{G}_{hp}(\xi, \eta)}{R-L} &= \frac{(L-a)(b-R)}{(b-a)(R-L)} + \frac{L-a}{b-a}l_0(\xi)l_0(\eta) + \frac{b-R}{b-a}l_1(\xi)l_1(\eta) \\ &\quad + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi, \eta).\end{aligned}\quad (17)$$

The endpoints of  $K_i$  can be parameterized using the element length  $H = R-L$  and a real parameter  $0 \leq t \leq 1$ , so that  $L = a$  for  $t = 0$  and  $R = b$  for  $t = 1$ :

$$L = (1-t)a + t(b-H), \quad (18)$$

$$R = (1-t)(a+H) + tb. \quad (19)$$

Use (18) and (19), define relative element length  $H_{\text{rel}}$  by

$$H_{\text{rel}} = \frac{H}{b-a},$$

and compute

$$\frac{L-a}{b-a} = \frac{t(b-a-H)}{b-a} = t(1-H_{\text{rel}}), \quad (20)$$

$$\frac{b-R}{b-a} = \frac{(1-t)(b-a-H)}{b-a} = (1-t)(1-H_{\text{rel}}), \quad (21)$$

$$\frac{(L-a)(b-R)}{(b-a)(R-L)} = \frac{t(1-t)(b-a-H)^2}{(b-a)H} = t(1-t)\frac{(1-H_{\text{rel}})^2}{H_{\text{rel}}}. \quad (22)$$

Substitute (20)–(22) into (17) to obtain

$$\begin{aligned}\frac{\hat{G}_{hp}(\xi, \eta)}{H} &= t(1-t)\frac{(1-H_{\text{rel}})^2}{H_{\text{rel}}} + t(1-H_{\text{rel}})l_0(\xi)l_0(\eta) \\ &\quad + (1-t)(1-H_{\text{rel}})l_1(\xi)l_1(\eta) + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi, \eta).\end{aligned}\quad (23)$$

Finally, use the identity

$$\hat{G}_{hp}^{p,B}(\xi, \eta) = t\hat{G}_{hp}^{p,B}(\xi, \eta) + (1-t)\hat{G}_{hp}^{p,B}(\xi, \eta),$$

substitute (16) into (23), and factor out  $l_0(\xi)l_0(\eta)$  and  $l_1(\xi)l_1(\eta)$ :

$$\begin{aligned} \frac{\hat{G}_{hp}(\xi, \eta)}{H} &= t(1-t)\frac{(1-H_{\text{rel}})^2}{H_{\text{rel}}} \\ &+ tl_0(\xi)l_0(\eta) \left[ 1 - H_{\text{rel}} + \frac{1}{2}l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \right] \\ &+ (1-t)l_1(\xi)l_1(\eta) \left[ 1 - H_{\text{rel}} + \frac{1}{2}l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \right]. \end{aligned} \quad (24)$$

Indeed, the value  $t(1-t)(1-H_{\text{rel}})^2/H_{\text{rel}}$  is nonnegative for all  $t \in [0, 1]$  as well as the values  $tl_0(\xi)l_0(\eta)$  and  $(1-t)l_1(\xi)l_1(\eta)$ , for all  $[\xi, \eta] \in \hat{K}^2$ . Hence, the discrete Green's function  $G_{hp}$  is nonnegative in  $K_i^2$  if both expressions in the square brackets in (24) are nonnegative. To see that they impose the same restriction on the relative element length  $H_{\text{rel}}$ , let us introduce Lemma 5.1:

**Lemma 5.1.** *It is*

$$\min_{[\xi, \eta] \in \hat{K}^2} l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) = \min_{[\xi, \eta] \in \hat{K}^2} l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta).$$

*Proof.* Using the definition of the functions  $\kappa_i$ , it is easy to see that  $\kappa_k(\xi) = \kappa_k(-\xi)$  for  $k$  even and  $\kappa_k(\xi) = -\kappa_k(-\xi)$  for  $k$  odd. Therefore,  $\kappa_k(\xi)\kappa_k(\eta) = \kappa_k(-\xi)\kappa_k(-\eta)$  for every  $k = 2, 3, \dots$ . Moreover,  $l_0(\xi) = l_1(-\xi)$ , which yields

$$\begin{aligned} \min_{[\xi, \eta] \in \hat{K}^2} l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) &= \min_{[\xi, \eta] \in \hat{K}^2} l_1(-\xi)l_1(-\eta) \sum_{k=2}^p \kappa_k(-\xi)\kappa_k(-\eta) \\ &= \min_{[\xi, \eta] \in \hat{K}^2} l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta). \end{aligned}$$

□

Relation (24) and Lemma 5.1 motivate the following definition:

**Definition 5.1.** *By critical relative element length  $H_{\text{rel}}^*$  corresponding to a polynomial degree  $p \geq 2$  we mean the value*

$$\begin{aligned} H_{\text{rel}}^*(p) &= 1 + \frac{1}{2} \min_{(\xi, \eta) \in \hat{K}^2} l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \\ &= 1 + \frac{1}{2} \min_{(\xi, \eta) \in \hat{K}^2} l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta). \end{aligned} \quad (25)$$

For  $p = 1$  we define  $H_{\text{rel}}^* = 1$ .

**Theorem 5.1.** *If  $a \leq L < R \leq b$  and*

$$\frac{R - L}{b - a} \leq H_{\text{rel}}^*(p), \quad (26)$$

*then the function  $\hat{G}_{hp}(\xi, \eta)$  defined by (15) is nonnegative for all  $[\xi, \eta] \in \hat{K}^2 = [-1, 1]^2$ .*

*Proof.* Apply (26) and the definition of  $H_{\text{rel}}^*(p)$  to infer

$$\begin{aligned} 1 - H_{\text{rel}} + \frac{1}{2}l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \\ \geq 1 - H_{\text{rel}}^*(p) + \frac{1}{2}l_1(\xi)l_1(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \geq 0 \quad \forall [\xi, \eta] \in \hat{K}^2. \end{aligned}$$

Similarly,

$$1 - H_{\text{rel}} + \frac{1}{2}l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \geq 0 \quad \forall [\xi, \eta] \in \hat{K}^2.$$

Thus, all terms in (24) are nonnegative and we can conclude that

$$\hat{G}_{hp}(\xi, \eta) \geq 0 \quad \text{for all } [\xi, \eta] \in \hat{K}^2.$$

□

## Computation of $H_{\text{rel}}^*(p)$

In Table 1 we list the values of  $H_{\text{rel}}^*(p)$  for  $p = 1, 2, \dots, 20$ .

$p$	$H_{\text{rel}}^*(p)$	$p$	$H_{\text{rel}}^*(p)$	$p$	$H_{\text{rel}}^*(p)$	$p$	$H_{\text{rel}}^*(p)$
1	1	6	1	11	0.953759	16	0.968695
2	1	7	0.935127	12	0.969485	17	0.967874
3	9/10	8	0.987060	13	0.959646	18	0.969629
4	1	9	0.945933	14	0.968378	19	0.970855
5	0.919731	10	0.973952	15	0.964221	20	0.970814

Table 1: Critical relative element length  $H_{\text{rel}}^*(p)$  for  $p = 1, 2, 3, \dots, 20$ .

The values of  $H_{\text{rel}}^*(p)$  for  $p = 1, 2, \dots, 100$  are plotted in Figure 3. While the values  $H_{\text{rel}}^*(p)$  for  $p = 1, 2, 3, 4$  could be calculated analytically, results for  $p \geq 5$  are numerical, obtained with high accuracy.

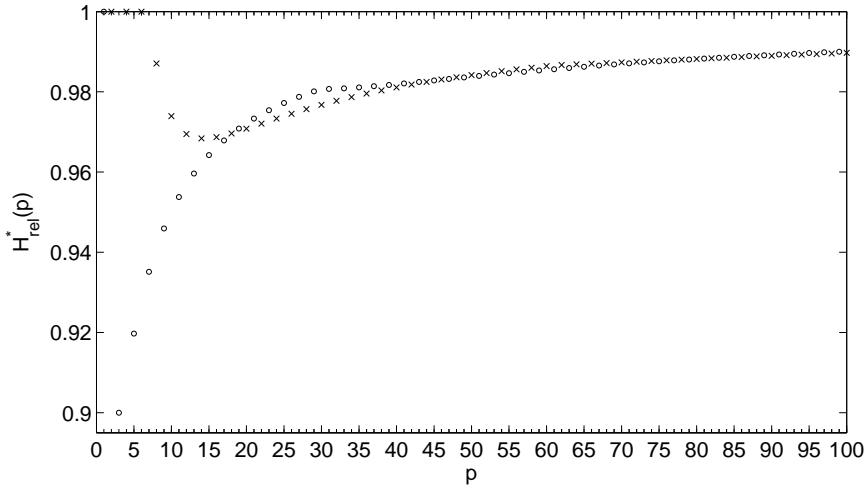


Figure 3: Critical relative element lengths  $H_{\text{rel}}^*(p)$  for  $p = 1, 2, \dots, 100$ . Circles indicate the values for  $p$  odd and crosses for  $p$  even.

## 6 Main Results

Let us summarize the conclusions of the previous analysis:

**Theorem 6.1.** *If the partition  $a = x_0 < x_1 < \dots < x_M = b$  of the domain  $\Omega = (a, b)$  satisfies the condition*

$$\frac{x_i - x_{i-1}}{b - a} \leq H_{\text{rel}}^*(p_i) \quad \text{for all } i = 1, 2, \dots, M, \quad (27)$$

where  $p_i \geq 1$  is the polynomial degree assigned to the element  $K_i = [x_{i-1}, x_i]$ , and  $H_{\text{rel}}^*(p_i)$  is defined by (25), then the problem (2) satisfies the discrete maximum principle (i.e.,  $u_{hp} \geq 0$  in  $\Omega$  for arbitrary  $f \in L^2(\Omega)$  which is nonnegative a.e. in  $\Omega$ ).

*Proof.* Let  $K_i$  be any element. By (15), condition (27), and Theorem 5.1 it holds

$$G_{hp}(x, z)|_{K_i^2} = G_{hp}(\xi, \eta) \geq 0 \quad \text{for all } [x, z] \in K_i^2.$$

Thus,  $G_{hp}(x, z) \geq 0$  in  $\bigcup_{i=1}^M K_i^2$ . Lemma 4.1 implies that  $G_{hp}(x, z) \geq 0$  also in  $\Omega^2 \setminus \bigcup_{i=1}^M K_i^2$ . Theorem 3.1 finishes the proof.  $\square$

Table 1 indicates that the restriction on the relative element length  $(x_i - x_{i-1})/(b - a)$  is strongest in the cubic case where  $H_{\text{rel}}^* = 9/10$ . Moreover, Figure 3 shows a steadily growing trend in  $H_{\text{rel}}^*$  for  $p \geq 50$ . These observations motivate the following conjecture:

**Conjecture 6.1.** *If the partition  $a = x_0 < x_1 < \dots < x_M = b$  of the domain  $\Omega = (a, b)$  satisfies the condition*

$$\frac{x_i - x_{i-1}}{b - a} \leq \frac{9}{10} \quad \text{for all } i = 1, 2, \dots, M,$$

*then the problem (2) satisfies the discrete maximum principle (i.e.,  $u_{hp} \geq 0$  in  $\Omega$  for arbitrary  $f \in L^2(\Omega)$  which is nonnegative a.e. in  $\Omega$ ).*

## Appendix

The Lobatto shape functions are defined by

$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x) dx, \quad j = 2, 3, \dots,$$

where  $P_j(x) = d^j/dx^j (x^2 - 1)^j / (2^j j!)$  stands for the  $j$ th-degree Legendre polynomial. The kernels are defined by  $\kappa_j(\xi) = l_j(\xi)/(l_0(\xi)l_1(\xi))$ , where  $l_0(\xi) = (1 - \xi)/2$ ,  $l_1(\xi) = (1 + \xi)/2$ , and  $\xi \in [-1, 1]$ . These kernels can be generated by the recurrence

$$\kappa_{j+2}(\xi) = \frac{\sqrt{2j+1}\sqrt{2j+3}}{j+2} \xi \kappa_{j+1}(\xi) - \frac{j-1}{j+2} \sqrt{\frac{2j+3}{2j-1}} \kappa_j(\xi), \quad j = 2, 3, \dots$$

For reference, we list several kernel functions  $\kappa_i$  (see, e.g., Section 3.1 in [14] or Section 1.2 in [12]):

$$\begin{aligned} \kappa_2(\xi) &= -\sqrt{6} \\ \kappa_3(\xi) &= -\sqrt{10}\xi \\ \kappa_4(\xi) &= -\frac{1}{4}\sqrt{14}(5\xi^2 - 1) \\ \kappa_5(\xi) &= -\frac{3}{4}\sqrt{2}(7\xi^2 - 3)\xi \\ \kappa_6(\xi) &= -\frac{1}{8}\sqrt{22}(21\xi^4 - 14\xi^2 + 1) \\ \kappa_7(\xi) &= -\frac{1}{8}\sqrt{26}(33\xi^4 - 30\xi^2 + 5)\xi \\ \kappa_8(\xi) &= -\frac{1}{64}\sqrt{30}(429\xi^6 - 495\xi^4 + 135\xi^2 - 5) \\ \kappa_9(\xi) &= -\frac{1}{64}\sqrt{34}(715\xi^6 - 1001\xi^4 + 385\xi^2 - 35)\xi \\ \kappa_{10}(\xi) &= -\frac{1}{128}\sqrt{38}(2431\xi^8 - 4004\xi^6 + 2002\xi^4 - 308\xi^2 + 7) \end{aligned}$$

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