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Abstract: *We continue our study of possible images of multilinear mappings. The case of bilinear mappings into* $\mathbf{R}^3$ *is well understood. The only possibilities are: (1) a subspace, (2) the exterior and boundary of a double elliptic cone, and (3) two lines through the origin together with the complement of the plane they span [1]. In this paper we obtain a general result which prevents a multilinear mapping into a finite dimensional target space from having an isolated subspace in its image. We obtain a sufficient condition for a multilinear mapping to have a subspace for its image. Finally we consider the special case of trilinear mappings* $F : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ *and obtain a complete list of possible images under the assumption that there is at least one associated bilinear mapping whose image is of type (3) listed above. In all there are four new possible image types to go with the three possibilities for the bilinear case.*

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1 Introduction

In this section we set up notation and work out some numerical examples. In Section 2 we obtain some more general results and examples. In Section 3 we describe all possible images of trilinear maps into $\mathbf{R}^3$ when there is at least one associated bilinear whose image is neither a subspace nor the exterior of a cone. All vector spaces considered in this paper are real.

The image type consisting of the complement of a plane together with two intersecting lines in the plane and through the origin occurs sufficiently often that we give it a name.

Definition 1.1. A *k-cricket* in $\mathbf{R}^3$ is a set of the form $(\mathbf{R}^3 \setminus P) \cup \bigcup_{i=1}^{k} \ell_i$, with P a plane in $\mathbf{R}^3$ which passes through the origin and $\ell_1, \ldots, \ell_k$ lines in P which intersect at the origin. If no confusion is likely, or we are allowing multiple values of k, we will simply use the term *cricket*. The lines ℓ_i will be called the *legs* of the cricket and the plane P the *associated plane*.

Rather than work through a specific example of, say, a 3-cricket we wait until Section 2 where Proposition 2.1 shows how to construct k-crickets for all $k \geq 2$.

The main result of [1] states, using definition 1.1, that the image of a bilinear mapping into $\mathbf{R}^3$ is a subspace, a 2-cricket or the boundary and exterior of a double elliptic cone. When $n > 2$ the description of the image of an n-linear mapping is more complicated. In [1, Theorem 2.6] we showed that the image of a bilinear mapping from $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^3$ is never the whole of $\mathbf{R}^3$. By contrast, the following example shows that a trilinear, and hence an n-linear mapping with $n > 2$ and two dimensional source spaces may map onto $\mathbf{R}^3$. This example is not too surprising in view of the very much larger domain.

Example 1.1. *Define* $F : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ *as follows:*

$$F((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (x_1 y_1 z_1, x_1 y_2 z_2, x_2 y_1 z_2)$$

Take $x_1 = 1 = y_1 = z_2$, $z_1 = a$, $y_2 = b$ and $x_2 = c$, to obtain $(a, b, c) \in \operatorname{range} F$, and $\operatorname{range} F = \mathbf{R}^3$.

We note in passing that if $(z_1, z_2) = (1, 1)$ we obtain a standard 2-cricket as in [1, Example 1.2]. Thus every point of $\mathbf{R}^3$ off the associated plane $x = 0$ is in the image of F. If, however, we let $z_1 = 0$ and $z_2 = y_1 = y_2 = 1$, then an arbitrary point $(0, x_1, x_2)$ is in the image, too. This gives an alternative demonstration that $\operatorname{range} F = \mathbf{R}^3$.

The next example is more interesting as it gives us a new type of image for a multilinear mapping into $\mathbf{R}^3$. With hindsight we can see it as a possibility arising from the union of images of two associated bilinear mappings both of whose images are crickets, but with different associated planes. In this case, clearly, the final image, if not the whole of $\mathbf{R}^3$, is the complement of a line through the origin together with the origin itself. This possibility is also maximal among images which are not the whole of $\mathbf{R}^3$.

Example 1.2. *Define the trilinear mapping* $F : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ *as follows:*

$$\begin{aligned} &F((x_1, x_2), (y_1, y_2), (z_1, z_2)) \\ &\quad = \quad (x_1y_1z_1 + x_2(y_1z_2 + y_2z_1), x_1y_1z_2 + x_2(-y_1z_1 + 2y_2z_2), x_1y_2z_1 + 2x_2y_2z_2) \end{aligned}$$

If we set $x_2 = 0$, the resulting associated bilinear mapping yields the standard 2-cricket whose legs are spanned by the vectors $(0, 0, 1)$ and $(0, 1, 0)$ and the associated plane is $x = 0$. For additional points in the image we must solve the simultaneous equations.

$$\begin{aligned} x_1y_1z_1 + x_2(y_1z_2 + y_2z_1) &= 0 \qquad (1) \\ x_1y_1z_2 + x_2(-y_1z_1 + 2y_2z_2) &= b \qquad (2) \\ x_1y_2z_1 + 2x_2y_2z_2 &= c. \qquad (3) \end{aligned}$$

A calculation, which we suppress, shows that if $b + c \neq 0$,

$$F(((3c - b)/2, -(b + c)/2), (1, 2c/(b + c)), (1, (c - b)/(b + c))) = (0, b, c).$$

Suppose now that $b+c = 0$. Repeating equation (1) and adding equations (2) and (3) leads us to

$$\begin{aligned} x_1y_1z_1 + x_2(y_1z_2 + y_2z_1) &= 0 \\ x_1(y_1z_2 + y_2z_1) + x_2(-y_1z_1 + 4y_2z_2) &= 0. \end{aligned}$$

For a solution with x_1 and x_2 not both zero we must have

$$0 = y_1z_1(-y_1z_1 + 4y_2z_2) - (y_1z_2 + y_2z_1)^2 = -(y_1z_1)^2 - (y_1z_2 - y_2z_1)^2.$$

This requires $y_1z_1 = 0 = y_1z_2 - y_2z_1$, which easily leads to $b = 0 = c$. Writing ℓ for the line $\langle(0, -1, -1)\rangle$, we see that the image of F is $(\mathbf{R}^3 \setminus \ell) \cup \mathbf{0}$.

Definition 1.2. An *all-but-a-line* in $\mathbf{R}^3$ is a set of the form $(\mathbf{R}^3 \setminus \ell) \cup \mathbf{0}$, where ℓ is a line in $\mathbf{R}^3$ which contains the origin.

Theorem 2.1 shows that we cannot go any further at guessing possible images without allowing for the possibility that an associated bilinear map has the exterior of a cone as its image. Indeed there are such associated bilinear maps in the all-but-a-line example 1.2. For another possibility suppose an associated bilinear map does have the exterior of a cone as its image. The complementary open cone intersects the plane associated with the cricket in an open bilateral sector. It is therefore somewhat natural to see as a possible image the complement of an open sector of the plane $x = 0$. More unexpectedly we find the complements of half open and closed sectors as well. Specifically, consider the following three subsets of $\mathbf{R}^3$.

1. $A = \{(0, b, c) : bc \neq 0 \text{ and } 0 \leq \alpha < b/c < \beta\}$
2. $B = \{(0, b, c) : bc \neq 0 \text{ and } 0 \leq \alpha \leq b/c < \beta\}$
3. $C = \{(0, b, c) : bc \neq 0 \text{ and } 0 \leq \alpha \leq b/c \leq \beta\}$

Examples 1.3, 1.4 and 1.5 will show that $\mathbf{R}^3 \setminus A$, $\mathbf{R}^3 \setminus B$ and $\mathbf{R}^3 \setminus C$ are all possible images of trilinear maps into $\mathbf{R}^3$. (The degenerate case $\alpha = \beta$ in C is, of course, an all-but-a-line.) These examples can all be obtained by perturbations of the same basic example which we proceed to set up.

Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ be defined by

$$F((x_1, x_2), (y_1, y_2), (z_1, z_2)) =$$
$$(x_1y_1z_1 + x_2(y_1z_2 + y_2z_1),\ x_1y_1z_2 + x_2(p_2y_1z_1 - 11y_2z_2),\ x_1y_2z_1 + x_2(p_3y_1z_1 + 3y_2z_2)).$$

With $x_1 = 1$ and $x_2 = 0$ we have

$$F((1, 0), (y_1, y_2), (z_1, z_2)) = (y_1z_1, y_1z_2, y_2z_1),$$

which gives us the standard cricket whose legs are the y and z-axes in the plane $x = 0$. To describe range F we must look for conditions for $(0, b, c) \in \operatorname{range} F$ when $bc \neq 0$. For this, if we have $y_1 = 0$ we must also have $x_2y_2z_1 = 0$. If $x_2y_2 = 0$ we have $b = 0$, while if $z_1 = 0$, we have $(0, b, c) \in \langle F(u_2, v_2, w_2)\rangle$. Except for this case we may take $y_1 = 1 = z_1$ and $y_2 = y$, $z_2 = z$. Now we seek consistency conditions for the set of equations.

$$\begin{aligned} x_1 + x_2(z + y) &= 0 \\ x_1z + x_2(p_2 - 11yz) &= b \\ x_1y + x_2(p_3 + 3yz) &= c. \end{aligned}$$

This requires $x_1 = -x_2(z + y)$ and hence

$$\begin{aligned} (-(z + y)z + p_2 - 11yz)x_2 &= b \\ (-(z + y)y + p_3 + 3yz)x_2 &= c. \end{aligned}$$

Writing $\theta = b/c$ we have the equivalent condition

$$-z^2 - 12yz + p_2 = \theta(-y^2 + 2yz + p_3) \neq 0. \tag{4}$$

Notice that if $-z^2 - 12yz + p_2 = 0 = (-y^2 + 2yz + p_3)$, then we have

$$0 = 2(z^2 + 12yz - p_2) + 12(y^2 - 2yz - p_3) = 2z^2 + 12y^2 - 2p_2 - 12p_3.$$

This is impossible if $p_2 + 6p_3 < 0$ and possible only if $y = 0 = z$, and hence $p_2 = 0 = p_3$, when $p_2 + 6p_3 = 0$. After completing the square we see that satisfying condition (4) requires us to first solve the equation.

$$(z + (6 + \theta)y)^2 = (4 + \theta)(9 + \theta)y^2 + p_2 - p_3\theta. \tag{5}$$

Example 1.3. *Use F as defined above with $p_2 = 0 = p_3$.*

With $x_1 = 1 = x_2$ we have

$$F((1, 1), (y_1, y_2), (z_1, z_2)) = (y_1z_1 + y_1z_2 + y_2z_1, y_1z_2 - 11y_2z_2, y_2z_1 + 3y_2z_2),$$

whose image, as can be checked by an argument similar to that in [1, Example 1.2], is the exterior of the cone, $(x + 2y - 12z)^2 + 100yz \geq 0$.

Since $p_2 = 0 = p_3$ we see that $(0, b, c) \in \operatorname{range} F$ if and only if there is a solution, with y and z not both zero, to the equation

$$(z + (6 + \theta)y)^2 = (4 + \theta)(9 + \theta)y^2.$$

Such solutions exist if and only if $(4+\theta)(9+\theta) \geq 0$. We note that $\theta = -11/3$ provides the case $(0, b, c) \in \langle F(u_2, v_2, w_2)\rangle$ which we bypassed above. Thus

$$\operatorname{range} F = \{\, (a, b, c) : a \neq 0, \text{ or } a = 0 \text{ and } b/c \notin (-9, -4) \,\}.$$

This gives us an example where the image is the complement of an open sector.

Example 1.4. *Use F as defined above, but with $p_2 = -9$ and $p_3 = 1$.*

Because $p_2 + 6p_3 = -3 < 0$ and $p_2 \neq 0$ we see that $(0, b, c) \in \operatorname{range} F$ if there is a solution to equation (5), which becomes

$$(z + (6 + \theta)y)^2 = (4 + \theta)(9 + \theta)y^2 - 9 - \theta.$$

This can be satisfied if and only if $\theta > -4$ (for large enough y) or $\theta \leq -9$ (for all y). This includes the bypassed case $\theta = -11/3$. We conclude that

$$\operatorname{range} F = \{\, (a, b, c) : a \neq 0, \text{ or } a = 0 \text{ and } b/c \notin (-9, 4] \,\}.$$

This is an example where the image is the complement of a half open sector.

For our final example in this series the image will be the complement of a closed sector.

Example 1.5. *Use F as defined above, but with $p_2 = -7$ and $p_3 = 0$.*

Just as in Example 1.4 we look for all solutions of equation (5) which becomes

$$(z + (6+\theta)y)^2 = (4+\theta)(9+\theta)y^2 - 7.$$

Necessary and sufficient conditions for solution are $\theta < -9$ or $\theta > -4$ (for large enough y). This includes the bypassed case $\theta = -11/3$. We conclude that

$$\operatorname{range} F = \{\, (a, b, c) : a \neq 0, \text{ or } a = 0 \text{ and } b/c \notin [-9, -4] \,\}.$$

We note for this example that any negative value for p_2 will work equally well, as will $p_3 = 1$ with $p_2 < -9$.

Our next example might be anticipated by taking the union of a cricket with the exterior of a cone when one of the cricket legs lies in the interior of the cone.

Definition 1.3. A *butterfly* in $\mathbf{R}^3$ is a set of the form $(\mathbf{R}^3 \setminus P) \cup B$ with $B \subset P$ and $B = S \cup \ell$ with S a closed bilateral sector in P whose vertex is at the origin and ℓ is a line through the origin in P in the complement of S.

Example 1.6. *Define the trilinear mapping $F : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ as follows:*

$$F((x_1, x_2), (y_1, y_2), (z_1, z_2)) \;=\; (x_1y_1z_1 + x_2y_1z_2, x_1y_1z_2 - x_2y_1z_1, x_1y_2z_1 + x_2(y_1z_1 + y_2z_2)).$$

With $x_1 = 1$ and $x_2 = 0$ we again have

$$F((1,0), (y_1, y_2), (z_1, z_2)) = (y_1z_1, y_1z_2, y_2z_1),$$

which again gives us the standard cricket whose legs are the y and z-axes in the plane $x = 0$. To describe $\operatorname{range} F$ we must look for conditions for $(0, b, c) \in \operatorname{range} F$ when $bc \neq 0$. We start with the case $y_1z_1 \neq 0$ and take $y_1 = 1 = z_1$, $y_2 = y$ and $z_2 = z$ and try to solve

$$\begin{aligned} x_1 + x_2z &= 0 \\ x_1z - x_2 &= b \\ x_1y + x_2(1 + yz) &= c. \end{aligned}$$

These equations lead to $x_2 = c$, $b = -x_2(z^2+1) = -c(1+z^2)$, and $x_1 = -x_2z = -cz$; while y can be arbitrary. These conditions are met if and only if $-b/c = 1 + z^2 \geq 1$. Thus range F contains all points in the open bilateral sector bounded by the lines $b + c = 0$ and $c = 0$, together with the line $b + c = 0$ itself. Now for the case when $y_1z_1 = 0$, we see that this also requires $x_2y_1z_2 = 0$ which leads immediately to $b = 0$ or $c = 0$.

We conclude that $\operatorname{range} F$ is a butterfly defined above with B the closed bilateral sector bounded by the lines $y + z = 0$ and $z = 0$ and ℓ is the z-axis.

2 Particular Cases

We now consider some general results for particular cases of multilinear maps into $\mathbf{R}^3$. Our first result is the promised general example of a k-cricket. The restriction to two dimensional source spaces is clearly unnecessary, but saves a lot of zeros.

Proposition 2.1. *Suppose that $V_1, \ldots, V_n$ are two dimensional spaces and*

$$\operatorname{rank}\begin{bmatrix} \alpha_1 & \ldots & \alpha_n \\ \beta_1 & \ldots & \beta_n \end{bmatrix} = 2. \tag{6}$$

If the n-linear mapping $F : V_1 \times \ldots \times V_n \to \mathbf{R}^3$ is defined by means of its component n-linear forms as follows

$$\begin{aligned} F_x((x_1, y_1), \ldots, (x_n, y_n)) &= x_1 \cdots x_n \\ F_y((x_1, y_1), \ldots, (x_n, y_n)) &= \sum_{i=1}^{n} \alpha_i x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_n \\ F_z((x_1, y_1), \ldots, (x_n, y_n)) &= \sum_{i=1}^{n} \beta_i x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_n, \end{aligned} \tag{7}$$

then range F *is a cricket whose associated plane P is described by the equation $x = 0$.*

Let F be n-linear and given by equations (7). We find range F. Suppose first that $x = 0$. For some value of i we have $x_i = 0$. Then

$$\begin{aligned} F_y((x_1, y_1), \ldots, (x_n, y_n)) &= \alpha_i x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_n \\ F_z((x_1, y_1), \ldots, (x_n, y_n)) &= \beta_i x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_n \end{aligned} \tag{8}$$

Thus $F((x_1, y_1), \ldots, (x_n, y_n))$ is parallel to the vector $(0, \alpha_i, \beta_i)$. It follows that the intersection of range F with the plane P is a set of k lines (corresponding to the distinct non-zero lines spanned by the vectors $(0, \alpha_i, \beta_i)$) all passing through the origin. Clearly $2 \leq k \leq n$.

If $x \neq 0$, every vector (x, y, z) is in the image of F. Indeed, in order to obtain a vector $(1, y', z')$, set $x_i = 1, i = 1, \ldots n$ and solve the system of linear equations

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i y_i &= y' \\ \sum_{i=1}^{n} \beta_i y_i &= z' \end{aligned}$$

for the variables y_i (condition (6) guarantees that the system is consistent). Next write $(x, y, z) = x(1, y/x, z/x)$ and use multilinearity of F. The image of F is a k-cricket as claimed.

We need an easy consequence of our work in [1].

Lemma 2.1. *Let U and V be two dimensional and $f : U \times V \to \mathbf{R}^3$ be a bilinear mapping whose image is not a subspace of $\mathbf{R}^3$. Then the image of f is a cricket if and only if there exist non zero vectors $u \in U$ and $v \in V$ such that $f(u, v) = 0$. In this case:*

1. *The legs of the cricket are the lines $\langle f(u, \cdot)\rangle$, and $\langle f(\cdot, v)\rangle$;*
2. *The associated plane is spanned by these two lines;*
3. *If $\langle f(x, y)\rangle$ is a leg of the cricket then exactly one of* $\operatorname{range} f(x, \cdot)$ *and* $\operatorname{range} f(\cdot, y)$ *has rank one.*

By [1, Theorem 2.6] we may choose bases in U, V and $\mathbf{R}^3$ such that, if $\operatorname{range} f$ is a cricket

$$f((x_1, x_2), (y_1, y_2)) = (x_1y_1, x_1y_2, x_2y_1),$$

and if not,

$$f((x_1, x_2), (y_1, y_2)) = (x_1y_1 + x_2y_2, x_1y_2, x_2y_1).$$

Suppose $u = (x_1, x_2)$, $v = (y_1, y_2)$ and $f(u, v) = 0$. Since $u \neq 0$ we may assume $x_2 \neq 0$. Then, in both cases above the third coordinate gives $y_1 = 0$ and in the second case the first coordinate then gives $y_2 = 0$ contradicting $v \neq 0$. Thus we have the first case, a cricket, and $y_2 \neq 0$ forces $x_1 = 0$. This proves 1, and 2 and 3 follow easily.

The following result is basic linear algebra. We will use it several times in this section.

Lemma 2.2. *Let V be an n-dimensional vector space and $v_1, \dots, v_n$ an independent subset of V. If $w_1, \dots, w_n$ are arbitrary elements of V, then $\lambda v_1 + w_1, \dots, \lambda v_n + w_n$ are independent for all but at most n values of λ.*

The easiest proof is to look at the eigenvalues of the linear map $A : V \to V$ defined by $Av_i = w_1$ for $i = 1, \dots, n$.

Our next result could be obtained from the results of Section 3. We include it here because the additional hypothesis allows for a very much more conceptual approach.

Theorem 2.1. *Suppose U, V and W are two dimensional, let $F : U \times V \times W \to \mathbf{R}^3$ be a trilinear mapping such that none of its associated bilinear maps has the exterior of a cone as its range. Then* $\operatorname{range} F$ *is either a subspace, an all-but-a-line or a k-cricket with $k = 2$ or $k = 3$.*

Suppose that range F is neither a subspace, a 2-cricket nor an all-but-a-line. By [1, Theorem 1.6] there is an associated bilinear mapping, say $F(u_1, \cdot, \cdot)$ whose image is a cricket. By Lemma 2.1 there exist $0 \neq v_2 \in V$ and $0 \neq w_2 \in W$ such that $F(u_1, v_2, w_2) = 0$. Then the legs of the cricket are range $F(u_1, v_2, \cdot)$ and range $F(u_1, \cdot, w_2)$ and the associated plane P is spanned by these two lines.

Since range F is not a 2-cricket there are $u_2 \in U$, $v \in V$ and $w \in W$ such that $F(u_2, v, w) \notin \text{range}\, F(u_1, \cdot, \cdot)$.

We first show that $v \notin \langle v_2 \rangle$ and $w \notin \langle w_2 \rangle$. If this were false we would have $F(u_2, v_2, w_2) \notin \text{range}\, F(u_1, \cdot, \cdot)$. We start by choosing $v \notin \langle v_2 \rangle$ and $w \notin \langle w_2 \rangle$ and note that $F(u_1, v, w)$, $F(u_1, v, w_2)$ and $F(u_1, v_2, w)$ are linearly independent elements of $\mathbf{R}^3$. By Lemma 2.2, $F(\lambda u_1 + u_2, v, w)$, $F(\lambda u_1 + u_2, v, w_2)$ and $F(\lambda u_1 + u_2, v_2, w)$ are linearly independent for all but at most three values of λ. The range of the associated bilinear map $F(\lambda u_1 + u_2, \cdot, \cdot)$ cannot be $\mathbf{R}^3$ ([1, Theorem 2.6]), is not the exterior of a cone, by hypothesis, and spans $\mathbf{R}^3$ for these values of λ. Accordingly it must be a cricket. Since $F(u_1, v_2, w_2) = 0$ we may replace u_2 by an appropriate $\lambda u_1 + u_2$ if necessary and assume that range $F(u_2, \cdot, \cdot)$ is a cricket. The associated plane for this cricket coincides with P or meets it in a line. If it met P in a line then range F would be $\mathbf{R}^3$ or an all-but-a-line. Hence the associated plane for the mapping $F(u_2, \cdot, \cdot)$ is P and $F(u_2, v_2, w_2)$ spans a leg. By Lemma 2.1 we may assume that $F(u_2, v_2, \cdot)$ has rank 1 so there exists $w_1 \notin \langle w_2 \rangle$ such that $F(u_2, v_2, w_1) = 0$. Now consider the associated linear map, $F(u_1 + u_2, v_2, \cdot)$. Its range contains $F(u_1 + u_2, v_2, w_2) = F(u_2, v_2, w_2)$ and $F(u_1 + u_2, v_2, w_1) = F(u_1, v_2, w_1)$ which are linearly independent and span P. Since range $F \neq \mathbf{R}^3$ this is impossible.

Thus we may assume that $F(u_1, v, w) \neq 0$.

If $F(u_1, v, w) \in P$, then P is spanned by the vectors $F(u_1, v, w)$ and $F(u_2, v, w)$ so range $F(\cdot, v, w) = P$ and range $F = \mathbf{R}^3$ a contradiction. Thus $F(u_1, v, w) \notin P$ and it follows from Lemma 2.1 that $v \notin \langle v_2 \rangle$ and $w \notin \langle w_2 \rangle$. Now observe that the three vectors $F(u_1, v, w)$, $F(u_1, v, w_2)$ and $F(u_2, v, w)$ are linearly independent, since range $F(\cdot, v, \cdot)$ cannot be $\mathbf{R}^3$ it must be a cricket. Arguing as above we conclude that the associated plane for the mapping $F(\cdot, v, \cdot)$ is P and the legs are spanned by $F(u_1, v, w_2)$ and $F(u_2, v, w)$. Since $F(u_1, v, w) \neq 0$, Lemma 2.1 shows that $F(u_2, v, w_2) = 0$. It follows similarly that $F(u_2, v_2, w) = 0$. Consider $F(u_1 + u_2, \cdot, \cdot)$. We have $F(u_1 + u_2, v, w_2) = F(u_1, v, w_2) \in P$ and $F(u_1 + u_2, v_2, w) = F(u_1, v_2, w) \in P$ and these two are linearly independent and span P. It follows that range $F(u_1 + u_2, \cdot, \cdot)$ is a cricket with associated plane P and legs the two vectors we just calculated. Since $F(u_1 + u_2, v, w) = F(u_1, v, w) + F(u_2, v, w)$ is the sum of an element outside P with an element from P it cannot be 0. Lemma 2.1 allows us to conclude $0 = F(u_1 + u_2, v_2, w_2) = F(u_2, v_2, w_2)$.

Because $F(u_2, v, w) \in P$ there exist constants α and β such that

$$F(u_1, v, w) = \alpha F(u_1, v, w_2) + \beta F(u_1, v_2, w).$$

This enables us to calculate as follows.

$$\begin{aligned} & F(x_1 u_1 + x_2 u_2, y_1 v + y_2 v_2, z_1 w + z_2 w_2) \\ & \quad = \; x_1 y_1 z_1 F(u_1, v, w) + (x_1 y_1 z_2 + \alpha x_2 y_1 z_1) F(u_2, v, w_2) \\ & \qquad + (x_1 y_2 z_1 + \beta x_2 y_1 z_1) F(u_1, v_2, w). \end{aligned}$$

Comparison with equations (7) shows that range F is a 3-cricket with legs $F(u_1, v, w_2)$, $F(u_1, v_2, w)$, $F(u_2, v, w)$. The legs can also be checked directly by consecutive substitutions $x_1 = 0$, $y_1 = 0$ and $z_1 = 0$.

Corollary 2.1. *If U, V and W are two dimensional and $F : U \times V \times W \to \mathbf{R}^3$ is a trilinear mapping into $\mathbf{R}^3$ whose range is a 3-cricket then there is a choice of bases in the various spaces such that F can be realized as a special case of Example 2.1 with matrix,*

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{bmatrix}$$

and $\alpha\beta \neq 0$.

Both the hypothesis of Theorem 2.1 and the working assumptions of the proof are satisfied, and the proof of the theorem leads directly to the required representation.

We conclude this section with two general theorems about multilinear mappings. The first contains a different proof of [1, Theorem 1.3] (the case $k = 1$), while it generalizes [1, Theorem 1.6] (which is the case $k = 2$).

Theorem 2.2. *Let k be a positive integer and suppose $n \geq k$. Suppose $F : V_1 \times \cdots \times V_n \to V$ is an n-linear mapping and the image of every k-linear mapping associated with F spans a subspace of V of dimension at most k, then the image of F also spans a subspace of V of dimension at most k.*

The proof is by induction on k. We suppose then that the theorem is true for all positive integers k' such that $k' < k$. Now we proceed by induction on n. The case $n = k$ is a tautology so we now assume the theorem is true for $k' < k$ and all n, and also for k and for m-linear mappings when $k \leq m \leq n - 1$. These assumptions guarantee that $1 \leq k < n$, and hence that $n \geq 2$.

Suppose first that $k > 1$ and the images of all associated $(k-1)$-linear mappings span subspaces of dimension at most $k - 1$. By the induction hypothesis the image of F spans a subspace of dimension at most $k - 1$.

We may therefore assume that there is an associated $(k-1)$-linear mapping whose image spans a subspace of dimension k. Without loss of generality we may assume that there exist $v_i \in V_i$ for $i = k, \ldots, n$ such that $\operatorname{range} F(\,\cdot\,, \ldots, \,\cdot\,, v_k, \cdots, v_n)$ spans a subspace of dimension k. We write P for this subspace. For $U = (u_1, \ldots, u_{k-1}) \in V_1 \times \cdots V_{k-1}$ we write

$$F(U, v_k, \ldots, v_n) = F(u_1, \ldots, u_{k-1}, v_k, \ldots, v_n).$$

Now we can say that there exist $U_1, \ldots, U_k \in V_1 \times \cdots V_{k-1}$ such that P is spanned by the k elements $p_i = F(U_i, v_k, \ldots, v_n)$, $i = 1, \ldots, k$. In case $k = 1$ we interpret this to mean $F(v_1, \ldots, v_n) \neq 0$ and take $P = \langle F(v_1, \ldots, v_n) \rangle$. With this interpretation the remainder of our argument applies equally to $k = 1$

Consider the $(n-1)$-linear mapping $F_n = F(\,\cdot\,, \cdots, v_n)$ from $V_1 \times \cdots \times V_{n-1} \to V$, defined by

$$F_n(w_1, \ldots, w_{n-1}) = F(\,\cdot\,, \ldots, v_n)(w_1, \ldots, w_{n-1}) = F(w_1, \ldots, w_{n-1}, v_n).$$

The image of each of its associated k-linear maps is the image of an associated k-linear map of F and hence spans a subspace of dimension at most k. The inductive hypothesis guarantees that $\operatorname{range} F_n$ spans a subspace of dimension at most k. Note that $p_i \in \operatorname{range} F_n$ for $i = 1, \ldots, k$ and conclude that

$$\operatorname{range} F_n \subseteq \langle \operatorname{range} F_n \rangle = P.$$

Similarly, with $F_{n-1} = F(\,\cdot\,, \cdots, v_{n-1}, \cdot)$ we have $\operatorname{range} F_{n-1} \subseteq P$. Choose $w_n \in V_n$, and $\lambda \in \mathbf{R}$. For $i = 1, \ldots, k$ we have

$$q_i = F(U_i, v_k, \ldots, v_{n-1}, v_n + \lambda w_n) \in \operatorname{range} F_{n-1} \subseteq P.$$

By Lemma 2.2 we may choose λ such that the q_i are linearly independent and hence

$$\langle q_1, \ldots, q_k \rangle = P.$$

Consider the $(n-1)$-linear mapping $F(\,\cdot\,, \cdots, v_n + \lambda w_n)$. Each of its associated k-linear mappings is an associated k-linear mapping of F so, by the inductive hypothesis again, its image spans a subspace of dimension at most k. Since $q_i \in \operatorname{range} F(\,\cdot\,, \cdots, v_n + \lambda w_n)$ for $i = 1, \ldots, n$ it follows that

$$\operatorname{range} F(\,\cdot\,, \cdots, v_n + \lambda w_n) \subseteq P.$$

Now let $w_i \in V_i$ for $i = 1, \ldots n-1$. We have

$$F(w_1, \ldots, w_n) = \lambda^{-1}\Big(F(w_1, \ldots, w_{n-1}, v_n + \lambda w_n) - F(w_1, \ldots, w_{n-1}, v_n)\Big) \in P - P = P.$$

Since w_n is an arbitrary element of V_n we have

$$\operatorname{range} F \subseteq P$$

as required.

Theorem 2.9 in [1] states that the image of any multilinear mapping into $\mathbf{R}^3$ is either a subspace or contains the exterior and a boundary of an elliptic cone or a 2-cricket. Therefore, for an arbitrary multilinear mapping into $\mathbf{R}^3$ the only peculiarities of the images can take place within the associated plane or inside the interior of an elliptic cone. The following result imposes some additional regularity by eliminating the possibility that the image could contain a subspace which is, in some sense, isolated from the remainder of the image. We deal throughout with finite dimensional spaces and assume that they carry the standard topology of $\mathbf{R}^p$ for some integer p.

Theorem 2.3. *Let* $F : V_1 \times V_2 \times \cdots \times V_n \to V$ *be multilinear and* $\dim V < \infty$. *Suppose there exists* $x \in \operatorname{range} F$*, an open ball* B_x *in* V*, and a subspace* S *of* V *such that*

$$x \in \operatorname{range} F \cap B_x \subseteq S,$$

then $\operatorname{range} F \subseteq S$.

Assume first that $\dim V_i \leq 2, i = 1, \ldots, n$. In the standard product topology of $V_1 \times \cdots \times V_n$ the multilinear mapping F is continuous, because each of its coordinate functions is linear. We proceed by induction on n.

If $n = 1$, F is linear, its image is a subspace of V and our assertion is clear. Let us assume that the result holds true for all k-linear mappings with $k < n$ and let F be n-linear. Let $x = F(v_1, \ldots, v_n)$ and take

$$(w_1, \ldots, w_n) \in V_1 \times V_2 \times \cdots \times V_n$$

By continuity of F, there is $\lambda > 0$ such that

$$(v_1 + \lambda w_1, \ldots, v_n + \lambda w_n) \in F^{-1}(B_x)$$

and

$$F(v_1 + \lambda w_1, \ldots, v_n + \lambda w_n) \in \operatorname{range} F \cap B_x \subseteq S.$$

By multilinearity of F we can write:

$$F(v_1 + \lambda w_1, \ldots, v_n + \lambda w_n) = F(v_1, \ldots, v_n) + L + \lambda^n F(w_1, \ldots, w_n)$$

where L is a linear combination of images of associated k-linear mappings ($k < n$) of F formed by fixing one or more of the arguments as a v_i. Each such associated k-linear mapping satisfies the hypotheses of the theorem, with the same x and the same B_x, so by the inductive hypothesis, its image is contained in S. Therefore, $L \in S$. Since also $x = F(v_1, \ldots, v_n) \in S$, we have that $F(w_1, \ldots, w_n) \in S$. Thus range $F \subseteq S$.

To remove the restriction on the dimensions of the spaces V_i suppose now that $x = F(v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n) \in V_1 \times \cdots \times V_n$. Replace each V_i with $\langle v_i, w_i \rangle$ and argue as above to see that $F(w_1, \ldots, w_n) \in S$.

In particular, if $\dim V = 3$ the image of F could not be made up of the exterior of a cone together with a finite number of lines in the interior of the cone or planes which intersect the interior of the cone.

3 Trilinear maps

In this section we assume that U, V and W are two dimensional, that $F : U \times V \times W \to \mathbf{R}^3$ is trilinear, that its image is not a subspace of $\mathbf{R}^3$ and that there is an associated bilinear mapping whose image is a cricket. Without loss of generality we may assume this image is associated with a vector $u_1 \in U$ and we write P for the plane associated with the 2-cricket range$(F(u_1, \cdot, \cdot))$.

By Lemma 2.1 there exist non zero $v_2 \in V$ and $w_2 \in W$ such that $F(u_1, v_2, w_2) = 0$. Our strategy now is to make appropriate simplifying choices of $u_2 \in U \setminus \langle u_1 \rangle$, $v_1 \in V \setminus \langle v_2 \rangle$ and $w_1 \in W \setminus \langle w_2 \rangle$. Then u_1 and u_2 form a basis for U, and similarly for V and W. We take $F(u_1, v_1, w_1)$, $F(u_1, v_1, w_2)$ and $F(u_1, v_2, w_1)$ as basis for $\mathbf{R}^3$. This makes the yz-plane in $\mathbf{R}^3$ the plane P associated with the image of $F(u_1, \cdot, \cdot)$. We note that $F(u_1, v_2, w_2) = 0$ and we write

$$\begin{aligned} F(u_2, v_1, w_1) &= (p_1, p_2, p_3) \\ F(u_2, v_1, w_2) &= (q_1, q_2, q_3) \\ F(u_2, v_2, w_1) &= (r_1, r_2, r_3) \\ F(u_2, v_2, w_2) &= (s_1, s_2, s_3). \end{aligned}$$

The first simplification is to replace, if necessary, u_2 with $u_2 - p_1 u_1$. This will enable us to assume that $p_1 = 0$.

We are then led to the following.

$$F((x_1, x_2), (y_1, y_2)(z_1, z_2)) = \begin{pmatrix} x_1y_1z_1 & & + x_2y_1z_2q_1 + x_2y_2z_1r_1 + x_2y_2z_2s_1 \\ x_1y_1z_2 & + x_2y_1z_1p_2 & + x_2y_1z_2q_2 + x_2y_2z_1r_2 + x_2y_2z_2s_2 \\ x_1y_2z_1 & + x_2y_1z_1p_3 & + x_2y_1z_2q_3 + x_2y_2z_1r_3 + x_2y_2z_2s_3 \end{pmatrix}. \tag{9}$$

To describe the image of F it is sufficient to determine those values of b and c with $bc \neq 0$ which can appear on the right hand side of (9). Thus we seek necessary and sufficient conditions for consistency of the following set of three nonlinear equations.

$$
\begin{aligned}
x_1y_1z_1 + x_2(y_1z_2q_1 + y_2z_1r_1 + y_2z_2s_1) &= 0 \\
x_1y_1z_2 + x_2(y_1z_1p_2 + y_1z_2q_2 + y_2z_1r_2 + y_2z_2s_2) &= b \\
x_1y_2z_1 + x_2(y_1z_1p_3 + y_1z_2q_3 + y_2z_1r_3 + y_2z_2s_3) &= c\,.
\end{aligned}
\tag{10}
$$

The core of our discussion will focus on solutions of equations (10) with $y_1 = 1 = z_1$ and with the notational simplifications $y_2 = y$, $z_2 = z$. This will lead to the somewhat simpler equations

$$
\begin{aligned}
x_1 + x_2(zq_1 + yr_1 + yzs_1) &= 0 \\
x_1z + x_2(p_2 + zq_2 + yr_2 + yzs_2) &= b \\
x_1y + x_2(p_3 + zq_3 + yr_3 + yzs_3) &= c\,.
\end{aligned}
\tag{11}
$$

Consistency of equations (11) is equivalent to being able to find y and z which satisfy the condition

$$c(-z(zq_1+yr_1+yzs_1)+p_2+zq_2+yr_2+yzs_2) = b(-y(zq_1+yr_1+yzs_1)+p_3+zq_3+yr_3+yzs_3) \neq 0.$$

Writing $\sigma_2 = s_2 - r_1$, and $\sigma_3 = s_3 - q_1$, this condition simplifies slightly to

$$c(-s_1yz^2-q_1z^2+\sigma_2yz+q_2z+r_2y+p_2) = b(-s_1y^2z-r_1y^2+\sigma_3yz+q_3z+r_3y+p_3) \neq 0 \tag{12}$$

The following lemma is needed in several situations.

Lemma 3.1. *If* $r_1 = 0$ *and* $\operatorname{range} F \neq \mathbf{R}^3$ *then* $r_2 = 0 = s_2 = \sigma_2$.

Since $r_1 = 0$, $F(u, v_2, w) \in P$ for all $u \in U$ and all $w \in W$. Consider the associated linear map $F(\,\cdot\,, v_2, w_1)$. Since $F(u_1, v_2, w_1) = (0, 0, 1)$ and $F(u_2, v_2, w_1) = (0, r_2, r_3)$ are both in P and $\operatorname{range} F \neq \mathbf{R}^3$, these vectors are linearly dependent, and hence $r_2 = 0$ as required. Now suppose that $r_3 \neq 0$. Arguing similarly with $F(u_2, v_2, w_1) = (0, 0, r_3)$ and $F(u_2, v_2, w_2) = (0, s_2, s_3)$ shows that $s_2 = 0$ and hence also $\sigma_2 = 0$. If $r_3 = 0$ we may replace u_2 with $u_2 + u_1$ to reach the same conclusion.

The ensuing discussion requires case by case consideration. The ultimate result, stated as Theorem 3.1, will be that the examples we have given exhaust all possible images in the case that F has an associated bilinear whose image is a 2-cricket.

Case 1. *If* $s_1 \neq 0$, *or if* $s_1 = 0 = r_1$ *and* $q_1\sigma_3 \neq 0$, *then* $\operatorname{range} F = \mathbf{R}^3$.

Put $z = ty$ and rewrite the first equation in condition (12) as

$$\frac{b}{c} = \frac{-s_1t^2y^3 - q_1t^2y^2 + \sigma_2ty^2 + q_2ty + r_2y + p_2}{-s_1ty^3 - r_1y^2 + \sigma_3ty^2 + q_3ty + r_3y + p_3}. \tag{13}$$

Write $h(t, y)$ for the ratio on the right hand side of equation (13). Assume $s_1 \neq 0$ and note that as $y \to \infty$ $h(t, y) \to t$ uniformly in t for t in any closed bounded interval which does not contain 0. For $bc \neq 0$ choose such an interval, say $[\alpha, \beta]$ with $\alpha < b/c < \beta$. Then choose y sufficiently large so that $h(\alpha, y) < b/c < h(\beta, y)$. An application of the intermediate value theorem shows that there exists $t \in (\alpha, \beta)$ such that $h(t, y) = b/c$. It is immediate that condition (12) is satisfied for these values of t and y so Case 1 is proved when $s_1 \neq 0$. If $s_1 = 0 = r_1$ and $q_1\sigma_3 \neq 0$, we have $\sigma_2 = 0$ by Lemma 3.1 and the proof is similar using $h(t, y) \to -q_1 t/\sigma_3$

We may proceed on the assumption that $s_1 = 0$, and hence that $F(u, v_2, w_2) \in P$ for all $u \in U$.

Case 2. *If* $s_1 = 0 = q_1 = r_1$, *then* $\operatorname{range} F$ *is* $\mathbf{R}^3$, *a 2-cricket or a 3-cricket.*

In this case we can bypass most of the arithmetic. Since $p_1 = 0 = q_1 = r_1 = s_1$ we see that the bilinear mapping $F(u_2, \cdot, \cdot)$ maps into the plane P. Since $\dim P = 2$, [1, Corollary 1.4] ensures that $\operatorname{range} F(u_2, \cdot, \cdot)$ is a subspace of P. If $\operatorname{range} F(u_2, \cdot, \cdot) = P$, then $\operatorname{range} F = \mathbf{R}^3$ and we are done. Otherwise we see that $\operatorname{range} F(u_2, \cdot, \cdot)$ is contained in a line, ℓ, in P.

Suppose now that $\operatorname{range} F$ is not the original 2-cricket so that there exist $b, c \neq 0$ such that $(0, b, c) = F(\alpha u_1 + \beta u_2, v, w) \in P$. Then $\alpha F(u_1, v, w) \in P$. If $\alpha F(u_1, v, w) \neq 0$, $\langle F(u_1, v, w)\rangle$ is one of the legs of our original 2-cricket. Since $bc \neq 0$, $\beta \neq 0$ and $\langle F(u_2, v, w)\rangle = \ell \neq \langle F(u_1, v, w)\rangle$. It follows that $P = \operatorname{range} F(\cdot, v, w)$, and again $\operatorname{range} F = \mathbf{R}^3$. If $\operatorname{range} F \neq \mathbf{R}^3$, then $\alpha F(u_1, v, w) = 0$ and $(0, b, c) \in \operatorname{range} F(u_2, \cdot, \cdot) = \ell$. Thus $\operatorname{range} F$ is a 3-cricket, whose legs are ℓ and the two legs of the 2-cricket $\operatorname{range} F(u_1, \cdot, \cdot)$. This completes the proof of Case 2.

Case 3. *Suppose* $s_1 = 0 = r_1$ *and* $q_1 \neq 0$, *then* $\operatorname{range} F$ *is* $\mathbf{R}^3$, *an all-but-a-line, a butterfly, the complement of an open sector or a 2-cricket.*

By Lemma 3.1 and Case 1 we may assume $s_2 = 0 = \sigma_2 = \sigma_3$. Equations (11) become

$$c(q_1 z^2 + q_2 z + p_2) = b(q_3 z + r_3 y + p_3) \neq 0. \tag{14}$$

If $r_3 \neq 0$ and $bc \neq 0$ we choose z so that $q_1 z^2 + q_2 z + p_2 \neq 0$ and solve for y to satisfy condition (14) and conclude that $\operatorname{range} F = \mathbf{R}^3$.

If $r_3 = 0$ but $q_3 \neq 0$ the ratio $R(z) = (q_1 z^2 + q_2 z + p_2)/(q_3 z + p_3)$ takes all real values, all real values except one or all real values outside a bounded open interval. In the first two of these cases $\operatorname{range} F$ is $\mathbf{R}^3$ or an all-but-a-line. In the last case we need to consider points in $\operatorname{range} F$ arising from the possibility that $y_1z_1 = 0$. Returning to equations (10) we see that we will also have $x_2y_1z_2 = 0$ and must satisfy

$$\begin{aligned} x_1y_1z_2 &= b \\ x_1y_2z_1 + x_2y_2z_2s_3 &= c \end{aligned}$$

which forces $bc = 0$. The case $c = 0$ corresponds to $R(z) = \pm\infty$ while the case $b = 0$ may or may not be a possible value of $R(z)$. If $b = 0$ is a possible value of $R(z)$ we see that range F is the complement of a bilateral open sector. Otherwise range F is a butterfly.

If $r_3 = 0 = q_3$ and $p_3 \neq 0$ the possible values for $R(z)$, as defined above, lie in a semi-infinite closed interval. Otherwise the analysis is the same and leads to the same conclusion that range F is the complement of an open sector or a butterfly.

Finally, if $r_3 = 0 = q_3 = p_3$, we get no solutions to condition (14) and range F is the original 2-cricket.

Before finishing our case by case discussion we need another concept. With $s_1 = 0$ and $q_1r_1 \neq 0$ we mostly deal with condition (12) by considering the associated equation.

$$c(-q_1z^2 + \sigma_2yz + q_2z + r_2y + p_2) = b(-r_1y^2 + \sigma_3yz + q_3z + r_3y + p_3). \quad (15)$$

and looking for solutions which also satisfy condition (12). Suppose first there is solution (y_0, z_0) of equation (15) which does not satisfy condition (12). In other words we have

$$\begin{aligned} -q_1z_0^2 + \sigma_2y_0z_0 + q_2z_0 + r_2y_0 + p_2 &= 0 \\ -r_1y_0^2 + \sigma_3yz + q_3z_0 + r_3y_0 + p_3 &= 0. \end{aligned}$$

We will call (y_0, z_0) a *pseudo solution* of condition (12).

Case 4. *If $s_1 = 0$, $q_1r_1 \neq 0$, and (y_0, z_0) is a pseudo solution of condition (12), then* range F *is $\mathbf{R}^3$, an all-but-a-line or the complement of a bilateral open sector in the yz-plane.*

Since $q_1 \neq 0$, the quadratic polynomial $-q_1z^2 + \sigma_2y_0z + q_2z + r_2y_0 + p_2$ factors as $(z - z_0)(-q_1z + \alpha)$.

If $\sigma_3y_0 + q_3 \neq 0$, we may factor $-r_1y_0^2 + \sigma_3y_0z + q_3z + r_3y_0 + p_3$ as $(\sigma_3y_0 + q_3)(z - z_0)$. It follows that the ratio

$$\frac{-q_1z^2 + \sigma_2y_0z + q_2z + r_2y_0 + p_2}{-r_1y_0^2 + \sigma_3y_0z + q_3z + r_3y_0 + p_3} = \frac{(z - z_0)(-q_1z + \alpha)}{(\sigma_3y_0 + q_3)(z - z_0)},$$

as a function of z takes all real values except one, $(-q_1z_0 + \alpha)/(\sigma_3y_0 + q_3)$. Since this value may come from the omitted case $y_1z_1 = 0$ (discussed after Condition (10)) we see that range F is $\mathbf{R}^3$ or an all-but-a-line.

If $\sigma_3 y_0 + q_3 = 0$, we also have $-r_1 y_0^2 + r_3 y_0 + p_3 = 0$, and

$$-r_1 y^2 + \sigma_3 yz + q_3 z + r_3 y + p_3 = (y - y_0)(\sigma_3 z - r_1(y + y_0) + r_3). \qquad (16)$$

Interchanging the roles of y_0 and z_0 in the argument with which we began the proof of Case (4) again gives range F as $\mathbf{R}^3$ or an all-but-a-line unless

$$-q_1 z^2 + \sigma_2 yz + q_2 z + r_2 y + p_2 = (z - z_0)(\sigma_2 y - q_1(z + z_0) + q_2). \qquad (17)$$

Choose z_2 such that $\sigma_2 y_0 - q_1(z_2 + z_0) + q_2 = 0$, and y_2 such that $\sigma_3 z_0 - r_1(y_2 + y_0) + r_3 = 0$. Then $\sigma_2 y - q_1(z + z_0) + q_2 = \sigma_2(y - y_0) - q_1(z - z_2)$, and $\sigma_3 z - r_1(y + y_0) + r_3 = \sigma_3(z - z_0) - r_1(y - y_2)$. Thus Condition (12) becomes

$$c(z - z_0)(\sigma_2(y - y_0) - q_1(z - z_2) = b(y - y_0)(\sigma_3(z - z_0) - r_1(y - y_2)) \neq 0. \qquad (18)$$

Suppose $z_2 \neq z_0$, then our arguments, above applied to the pseudo solution (y_0, z_1) show that range F is $\mathbf{R}^3$ or an all-but-a-line, unless $z - z_2$ is a factor of the left side expression in (18). In that case we have $\sigma_2 = 0$ and (18) becomes

$$cq_1(z - z_0)(z - z_2) = b(y - y_0)(\sigma_3(z - z_0) - r_1(y - y_2)) \neq 0. \qquad (19)$$

Since $z_0 \neq z_2$ and $bcq_1 r_1 \neq 0$ we may choose z such that $-cq_1(z - z_0)(z - z_2)$ is not zero and has the same sign as $-br_1$, and hence is in the range of the quadratic (in y) $(y - y_0)(\sigma_3(z - z_0) - r_1(y - y_2))$. This shows that range $F = \mathbf{R}^3$.

The same argument works if $y_2 \neq y_0$ so we are left with the possibility that $y_1 = y_0$ and $z_1 = z_0$.

Condition (18) becomes

$$c(z - z_0)(\sigma_2(y - y_0) - q_1(z - z_0)) = b(y - y_0)(-r_1(y - y_0) + \sigma_3(z - z_0)) \neq 0.$$

If $\sigma_2 \sigma_3 = q_1 r_1$, we have $k \neq 0$ such that

$$\frac{(y - y_0)(-r_1(y - y_0) + \sigma_3(z - z_0))}{(z - z_0)(\sigma_2(y - y_0) - q_1(z - z_0))} = k\frac{y - y_0}{z - z_0}$$

unless $(y - y_0)/(z - z_0) = q_1/\sigma_2$. We conclude easily that range F is $\mathbf{R}^3$ or an all-but-a-line.

When $\sigma_2 \sigma_3 \neq q_1 r_1$ we simplify slightly by putting $Y = y - y_0$ and $Z = z - z_0$ and note that the only solution of

$$cZ(\sigma_2 Y - q_1 Z) = 0 = bY(-r_1 Y + \sigma_3 Z)$$

is the trivial $Y = 0 = Z$. Set $Y = tZ$ with $Z \neq 0$ The quadratic equation

$$r_1 bt^2 + (\sigma_2 c - \sigma_3 b)t - q_1 c = 0$$

has real solutions if and only if

$$(\sigma_2 c - \sigma_3 b)^2 + 4q_1 r_1 bc \geq 0. \tag{20}$$

In this case we see that range F contains, at least, all points in the yz-plane which lie outside an open sector. We must again check on solutions with $y_1 z_1 = 0$. Equations (10) become

$$\begin{aligned} x_1 y_1 z_1 + x_2(y_1 z_2 q_1 + y_2 z_1 r_1) &= 0 \\ x_1 y_1 z_2 + x_2(y_1 z_1 p_2 + y_1 z_2 q_2 + y_2 z_1 r_2 + y_2 z_2 s_2) &= b \\ x_1 y_2 z_1 + x_2(y_1 z_1 p_3 + y_1 z_2 q_3 + y_2 z_1 r_3 + y_2 z_2 s_3) &= c\,. \end{aligned}$$

A solution with $y_1 = 0$ requires $x_2 y_2 z_1 = 0$. Then $x_2 y_2 = 0$ forces $bc = 0$ and $z_1 = 0$ makes $(0, b, c)$ a multiple of $(0, s_2, s_3)$. The same holds when we start from $z_1 = 0$. Condition (20) is trivially satisfied if $bc = 0$. In the remaining case we have

$$(\sigma_2 s_3 - \sigma_3 s_2)^2 + 4q_1 r_1 s_2 s_3 = (r_1 s_3 + q_1 s_2)^2 \geq 0.$$

This shows that all points in range F are found from the case $y_1 z_1 \neq 0$. This completes the proof of Case (4).

The only remaining possibility is that $s_1 = 0$, $q_1 r_1 \neq 0$ and there are no pseudo solutions to Condition (12). We will need to consider two different ways in which this can happen and before dealing with these cases we need some preliminaries common to both.

We may now assume that any solution to equation (15) also satisfies condition (12).

We put $\theta = b/c$ and rewrite equation (15) as

$$q_1 z^2 - (\sigma_2 - \sigma_3\theta)yz - (q_2 - q_3\theta)z = r_1\theta y^2 + (r_2 - r_3\theta)y + p_2 - p_3\theta.$$

Without loss of generality we may assume that $q_1 = 1$. Temporarily write $P = p_2 - p_3\theta$ and similarly for Q, R and Σ, then complete the square to obtain the equivalent equation

$$(z - \Sigma y/2 - Q/2)^2 = (\Sigma^2/4 + r_1\theta)y^2 + (R - Q\Sigma/2)y + Q^2/4 + P \tag{21}$$

We can find y and z to satisfy equation (21) if and only if we can choose y so that the expression $E(y)$ on the right hand side of equation (21) is non negative.

Note that $E(y)$ can be non negative for some value of y if

1. $\Sigma^2/4 + r_1\theta > 0$ or

2. $\Sigma^2/4 + r_1\theta = 0$ and $R - Q\Sigma/2 \neq 0$ or

3. $\Sigma^2/4 + r_1\theta = 0$, $R - Q\Sigma/2 = 0$ and $Q^2/4 + P \geq 0$ or

4. $\Sigma^2/4 + r_1\theta < 0$ and $\Delta = (R - Q\Sigma/2)^2 - 4(Q^2/4 + P)(\Sigma^2/4 + r_1\theta) \geq 0$.

Since $r_1 \neq 0$, the closure, in the extended real line, of the set $\Sigma^2/4 + r_1\theta > 0$ contains 0 and at least one of $\pm\infty$. Thus $(0, 0, 1)$ and $(0, 1, 0)$ are in the closure of the subset of range F determined by $y_1 z_1 \neq 0$.

Case 5. *If $s_1 = 0$, $q_1 r_1 \sigma_3 \neq 0$ and there are no pseudo solutions to Condition (12) then* range F *is* $\mathbf{R}^3$*, an all-but-a-line or the complement of a bilateral sector.*

Observe first that, because $\sigma_3 \neq 0$, the quadratic expression in θ, $\Sigma^2/4 + r_1\theta$ is concave up. Suppose $\Sigma^2/4 + r_1\theta = 0$ for at most one value of θ. Then $\Sigma^2/4 + r_1\theta$ is non negative definite and we may choose y to make $E(y) \geq 0$ for all except perhaps one value of θ. In this case range F is $\mathbf{R}^3$ or an all-but-a-line.

We may continue on the assumption that there exist α and β with $\alpha < \beta$ such that $0 \notin (\alpha, \beta)$ and $\Sigma^2/4 + r_1\theta = 0$ for $\theta = \alpha$ and $\theta = \beta$. Concavity shows that $\Sigma^2/4 + r_1\theta > 0$ and, by item 1 above, that $(0, b, c) \in \text{range}\, F$ if $\theta \notin [\alpha, \beta]$. We also note that $\Sigma^2/4 + r_1\theta < 0$ for $\theta \in (\alpha, \beta)$ from which it follows that $0 \notin (\alpha, \beta)$

Now consider $\Delta = \Delta(\theta)$ on the interval (α, β). Expanding we have

$$\Delta = RQ\Sigma - r_1\theta Q^2 + P\Sigma^2 + R^2 - 4r_1 P\theta.$$

We see that Δ is a polynomial in θ of degree at most three, and that $\Delta(\alpha) \geq 0$ and $\Delta(\beta) \geq 0$.

Suppose first that $\Delta \geq 0$ on (α, β), then item 4 above shows that range F contains all of the plane $x = 0$ except perhaps for the two lines determined by $\theta = \alpha$ and $\theta = \beta$. We note that $\Delta(\alpha) = (R - Q\Sigma/2)(\alpha)^2 \geq 0$ so item 2 above puts $\theta = \alpha$ into range F unless $(R - Q\Sigma/2)(\alpha) = 0$. This shows that range F is $\mathbf{R}^3$ or an all-but-a-line unless $R - Q\Sigma/2 = 0$ at α and β. For this to happen $R - Q\Sigma/2$ must be a multiple of $\Sigma^2/4 + r_1\theta$. For the moment write $\Sigma^2/4 + r_1\theta = a$, $R - Q\Sigma/2 = ka$ and $b = Q^2/4 + P$. Then $\Delta = (ka)^2 - 4ab = a(k^2 a - 4b)$. By hypothesis, $a < 0$ immediately to the right of α. Since $\Delta \geq 0$ on (α, β) we have $k^2 a - 4b \leq 0$ immediately to the right of α, so continuity gives us $b(\alpha) = \lim -(k^2 a - 4b)/4 \geq 0$, and item 3 puts $\theta = \alpha$ into range F. The same argument holds at β so we may conclude that range F is $\mathbf{R}^3$.

The second alternative is that $\Delta < 0$ at some point of the interval (α, β). In this case a degree argument shows that there exist γ and δ such that $\alpha \leq \gamma < \delta \leq \beta$ and

$$(\alpha, \beta) \cap (\Delta < 0) = (\gamma, \delta).$$

It follows that $(0, b, c) \notin \operatorname{range} F$ when $\theta \in (\gamma, \delta)$ and $(0, b, c)$ may or may not be in $\operatorname{range} F$ when θ is one of γ or δ.

We now need to determine elements of $\operatorname{range} F$ arising from $y_1 z_1 = 0$. The argument at the end of Case 4 shows that these satisfy $bc = 0$ or are multiples of $(0, s_2, s_3)$. We have already seen that $(0, 1, 0)$ is outside, and $(0, 0, 1)$ is, at worst, on the boundary of the sector defined by $\theta \in (\gamma, \delta)$. We need to consider the possibility that $s_2 s_3 \neq 0$ and $\theta = s_2/s_3$. If this is the case we have

$$\Sigma^2/4 + r_1\theta = (s_2 - r_1 - (s_3 - 1)s_2/s_3)^2/4 + r_1 s_2/s_3 = (r_1 + s_2/s_3)^2/4 \geq 0.$$

It follows that $s_2/s_3 \notin (\gamma, \delta)$. Thus we may safely conclude that $\operatorname{range} F$ is the complement of the bilateral sector determined by $\theta \in [\gamma, \delta]$ together with both, either or neither of the boundary lines.

Case 6. *If $s_1 = 0 = \sigma_3$, $q_1 r_1 \neq 0$ and there are no pseudo solutions to Condition (12) then* $\operatorname{range} F$ *is $\mathbf{R}^3$, an all-but-a-line or the complement of a bilateral sector.*

In this case $\Sigma^2 + r_1\theta = r_1\theta + \sigma_2{}^2$ and its negative set excludes 0 and includes exactly one of $\pm\infty$. If Δ is cubic the coefficient of θ^3 is $-r_1 q_3{}^2$ whose sign is opposite to r_1. Thus $\Delta > 0$ as θ approaches the infinite end of the negative set of $\Sigma^2 + r_1\theta$. Since $\Delta \geq 0$ at the finite end, it follows that the set on which both $\Sigma^2 + r_1\theta$ and Δ are negative is empty or a finite open interval. We can conclude as in Case 5 above.

If Δ is not cubic, it is at most quadratic and its negative set is empty or a single open interval which may be semi-infinite. The intersection of this negative set with the negative set of $\Sigma^2 + r_1\theta$ is an open interval and again the argument of Case 5 applies. We note that when the intersection of the negative sets is semi-infinite the infinite end does correspond to a point in $\operatorname{range} F$.

We may summarize the last six cases in the following theorem.

Theorem 3.1. *Suppose U, V and W are two dimensional and that $F : U \times V \times W \to \mathbf{R}^3$ is a trilinear mapping. If there is a bilinear mapping associated with F whose image is a cricket, then* $\operatorname{range} F$ *is one of the following, a 2-cricket, a 3-cricket, an all-but-a-line, a butterfly, or the complement of a sector.*

4 Afterword

Our results leave many questions unanswered. We have not, as yet, been able to provide a clear description of trilinear images when there are no associated bilinears whose images are crickets. We do know from [1, Theorem 1.6] that if all associated bilinears have a subspace of dimension at most two for their image then so does the trilinear. It is not hard to see that if no associated bilinears have crickets as images and at least one has the complement of a cone the complete description of its image will require consideration of fourth degree polynomials as consistency conditions. In view of the somewhat outlandish possibilities we have exposed above it will be very interesting indeed to see what can happen in this last case.

Theorem 2.1 raises an interesting question. We do not have an example of a trilinear mapping from two dimensional source spaces into three dimensions whose range is an all-but-a-line and which has the property that no associated bilinear mapping has the exterior of a cone as its image. Our question is whether the prohibition of an all-but-a-line is necessary for Theorem 2.1, or is in fact a consequence of the hypothesis that there are no associated bilinears whose images are cone exteriors.

It is always possible to argue that the correct way to approach the determination of the image of a multilinear mapping is to use the canonical factorization through the tensor product. In the case of bilinear mappings we were able to go a short distance in this direction. The tensors $u \otimes v$ are easily characterized in $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ as elements (x_1, x_2, x_3, x_4) of $\mathbf{R}^4$ which satisfy $x_1 x_4 = x_2 x_3$. In [1] we did not use linear images of the set of tensors to characterize images of bilinear mappings. We did show, in Remark 2.7, that two simple projections of $\mathbf{R}^4 \to \mathbf{R}^3$ gave the two possible non subspace images of bilinear mappings into $\mathbf{R}^3$. For trilinears we have no similar result. Just to characterize tensors $u \otimes v \otimes w$ in $\mathbf{R}^2 \bigotimes \mathbf{R}^2 \bigotimes \mathbf{R}^2$ requires eight conditions of the type $u_2 u_7 = u_3 u_6$. We were unable to find a way to use this characterization to describe linear images of this set of tensors so as to simplify any of our arguments. Nor have we looked for projections $\mathbf{R}^8 \to \mathbf{R}^3$ which map the set of tensors to the various non subspace images we have identified so far. (It would be easy to adapt [1, Remark 1] to produce a 2-cricket.)

Another possible analogy with the bilinear situation would be a result like [1, Theorem 2.5] whereby if $F : U \times V \times W \to \mathbf{R}^3$ is trilinear and its image is not a subspace of $\mathbf{R}^3$ there are two dimensional subspaces U_1, V_1 and W_1 such that F restricted to $U_1 \times V_1 \times W_1$ has the same image. We have been unable to obtain a result like this even allowing the subspaces to be three-dimensional. One can, of course, look for an even closer analogy and try for complementary subspaces such that, for example, $U = U_1 \oplus L$ and F annihilates $L \times V \times W$.

References

[1] S. J. Bernau, P. J. Wojciechowski, *Images of bilinear mappings into* $\mathbf{R}^3$,Proc. of the American Mathematical Society, 124(12) 3605-3612 (1996).