The University of Texas at El Paso
Department of Mathematical Sciences
Research Reports Series

El Paso, Texas
Research Report No. 2006-08

Continuous *hp* Finite Elements Based on Generalized Eigenfunctions

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Abstract: *In this paper we present a new class of hp finite elements for product and simplicial geometries in* $\mathbb{R}^d$ *based on generalized eigenfunctions of the Laplace operator. Due to simultaneous orthogonality of the generalized eigenfunctions under both the* H^1_0 *and* L^2 *products, such finite elements have outstanding conditioning properties for second order elliptic problems. Analysis is accompanied by numerical examples, including comparisons to other popular sets of higher-order shape functions for both hp and spectral elements.*

AMS subject classification: 35B50, 65N60

Keywords: *hp*-FEM, optimal shape functions, elliptic problems, generalized eigenfunctions, Laplace operator

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Acknowledgment

The first author gratefully acknowledges the financial support of the U.S. Department of Defense under Grant No. 05PR07548-00, of the NSF under Grant No. DMS-0532645, and of the Grant Agency of the Czech Republic under Grant No. 102-05-0629. The second author was supported in part by the Grant Agency of the Czech Republic, project No. 201/04/P021 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

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1 Introduction

The p-version of the Finite element method (p-FEM) was first used and studied systematically in the mid 1970s by B. Szabó, I. Babuška et. al. (see, e.g., [4, 14, 15, 16, 17]). It was discovered during the 1980s (see [2] and the references therein) that the choice of higher-order shape functions has a dramatical effect on the conditioning of the discrete problems. Since then, significant amount of work has been devoted to the improvement of higher-order shape functions on various types of finite elements (see, e.g., [1, 8, 12, 19, 22, 23]). The search for optimality continues until today.

In particular, integrated Legendre polynomials (sometimes also called *Lobatto shape functions*) became a prominent tool for the design of higher-order elements for second-order elliptic PDEs due to their orthogonality under the H^1_0-product (L^2-product of first derivatives). This orthogonality avoids off-diagonal contributions of higher-order shape functions to the stiffness matrix of the Laplace operator in one spatial dimension. In higher spatial dimension, shape functions based on the integrated Legendre polynomials proved to have better conditioning than other, older types of higher-order shape functions [1, 22]. However, the Lobatto shape functions are not orthogonal in the L^2-product and their orthogonality in the H^1_0-product does not naturally extend to Cartesian product elements in $\mathbb{R}^d$.

We show that these drawbacks can be avoided by replacing the integrated Legendre polynomials by the generalized eigenfunctions of the Laplacian. In Section 2 we recall the generalized eigenfunctions of partial differential operators and discuss some of their properties on the general level. In Section 3 we apply these results to the Laplace operator in one spatial dimension and compare the generalized eigenfunctions to the integrated Legendre polynomials. In Section 4 we use the one-dimensional generalized eigenfunctions to construct higher-order shape functions for Cartesian product elements in $\mathbb{R}^d$. In Section 5 we construct new higher-order shape functions for simplicial elements in $\mathbb{R}^d$ by solving the corresponding generalized eigenproblem. Analytical results are illustrated by numerical experiments throughout the text and additional numerical examples and comparisons are provided in Section 6.

Integrated Legendre polynomials

Before we proceed, let us recall briefly the integrated Legendre polynomials

$$l_i(x) = \int_{-1}^{x} L_{i-1}(\xi)\,\mathrm{d}\xi, \qquad i = 2, 3, \ldots \tag{1}$$

Obviously, these functions vanish at the endpoints of the interval $K_a = (-1, 1)$, and therefore they are used as *bubble functions (interior modes)* to extend the standard basis for piecewise-linear elements to higher-order approximations [21]. It also follows from (1) that

$$a(l_i, l_j) = (l_i, l_j)_{H^1_0(K_a)} = \int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}. \tag{2}$$

It can easily be verified that shape functions (1) are not orthogonal in the L^2-product,

$$b(l_i, l_j) = (l_i, l_j)_{L^2(K_a)} = \int_{-1}^{1} l_i(\xi) l_j(\xi)\, \mathrm{d}\xi. \tag{3}$$

Consider for a moment a product quadrilateral element $K_q = (-1, 1)^2$. Polynomials (1) are used frequently to construct product shape functions on K_q in the form

$$\omega_{rs}(x_1, x_2) = l_r(x_1) l_s(x_2). \tag{4}$$

However, in this case it is easy to check that

$$\begin{aligned} (\omega_{ij}, \omega_{kl})_{H_0^1(K_q)} &= \int_{K_q} \nabla\omega_{ij}(x_1, x_2) \cdot \nabla\omega_{kl}(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \delta_{ik} \int_{-1}^{1} l_j(x) l_l(x) \mathrm{d}x + \delta_{jl} \int_{-1}^{1} l_i(x) l_k(x) \mathrm{d}x, \end{aligned}$$

which means that the product shape functions (4) do not inherit the H_0^1-orthogonality from the shape functions (1). We will see in the following that when this lack of orthogonality is cured, higher-order finite elements of remarkably better properties can be constructed.

2 Generalized Eigenfunctions

In this section we recall some properties of generalized eigenfunctions for later reference. We assume a bounded, simply connected domain $\hat{K} \subset \mathbb{R}^d$ with Lipschitz continuous boundary. The domain $\hat{K}$ is equipped with a space $\hat{W} = P_0^p(\hat{K})$, $\dim(\hat{W}) = n$ of polynomials vanishing on the boundary ∂K. Let $a(\cdot, \cdot) : \hat{W} \times \hat{W} \to \mathbb{R}$ be a bilinear form which appears in the weak formulation of a partial differential equation $Lu = f$. By *generalized eigenfunctions* of the operator L in the space $\hat{W}$ we mean polynomials $\psi_1, \psi_2, \ldots, \psi_n$ solving the following weak eigenproblem:

Find $(\psi_m, \lambda_m) \in \hat{W} \times \mathbb{C}$, $m = 1, 2, \ldots, n$ such that the functions ψ_1, ψ_2, …, ψ_m are linearly independent and moreover

$$a(\psi_m, v) = \lambda_m b(\psi_m, v) \qquad \text{for all } v \in \hat{W}. \tag{5}$$

Here, the bilinear form $b : \hat{W} \times \hat{W} \to \mathbb{R}$,

$$b(u, v) = \int_{\hat{K}} u(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \tag{6}$$

represents the L^2-product in the space $\hat{W}$.

By expanding the unknown functions ψ_m in terms of an arbitrary basis $\mathcal{B}_p = \{g_1, g_2, \ldots, g_n\}$ of the space $\hat{W}$,

$$\psi_k(\boldsymbol{x}) = \sum_{j=1}^{n} y_{jk} g_j(\boldsymbol{x}), \tag{7}$$

the weak eigenproblem (5) attains an equivalent matrix form [9, 13, 27],

$$SY = MY\Lambda. \tag{8}$$

Here, S is the stiffness matrix of the type $n \times n$, $s_{ij} = \int_{\hat{K}} g_j' g_i' \mathrm{d}x$, and M is the mass matrix, $m_{ij} = \int_{K_a} g_i g_j \mathrm{d}x$. The $n \times n$ matrix Y contains the unknown expansion coefficients y_{jk} in its columns. The diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ contains the corresponding eigenvalues. By (6), the matrix M is symmetric.

Theorem 2.1. *Let the matrices S and M be symmetric positive definite. Then problem (8) has a solution which is unique up to invariant subspaces corresponding to repeated eigenvalues.*

Proof. See proof of Theorem 8.7.1 of [9]. □

The proof of the following Lemma 2.1 is left to the reader as an easy exercise:

Lemma 2.1. *The solution (ψ_m, λ_m), $m = 1, 2, \ldots, n$ of (5) is invariant under the choice of the basis $\mathcal{B}_p$.*

In the important symmetric case it holds:

Lemma 2.2. *Let the bilinear form $a(\cdot,\cdot)$ in (5) be symmetric positive definite, and let (ψ_m, λ_m), $m = 1, 2, \ldots, p-1$ be the solution to (5). Whenever $\lambda_i \neq \lambda_j$, we have*

$$a(\psi_i, \psi_j) = \delta_{ij}, \qquad 1 \le i, j \le n. \tag{9}$$

If $\lambda_{i_1} = \lambda_{i_2} = \ldots = \lambda_{i_m}$, then the functions $\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_m}$ generate a subspace in $\hat{W}$ where one can find a basis satisfying (9). Moreover, we have

$$b(\psi_i, \psi_j) = \frac{\delta_{ij}}{\lambda_i}, \qquad 1 \le i, j \le n. \tag{10}$$

Proof. Let $a(\psi_i, v) = \lambda_i b(\psi_i, v)$ for all $v \in \hat{W}$ and $a(\psi_j, \tilde{v}) = \lambda_j b(\psi_j, \tilde{v})$ for all $\tilde{v} \in \hat{W}$. Putting ψ_i for $\tilde{v}$ and ψ_j for v, and using the symmetry of the form $a(\cdot,\cdot)$, we obtain

$$(\lambda_i - \lambda_j) b(\psi_i, \psi_j) = 0.$$

If $\lambda_i \neq \lambda_j$, then $b(\psi_i, \psi_j) = 0$ and it follows from (5) that $a(\psi_i, \psi_j) = 0$. The rest is obvious. □

On the computational side, the generalized eigenproblem (8) can be solved using standard mathematical software such as LAPACK or Matlab (in fact, the latter uses a function of the former for that purpose).

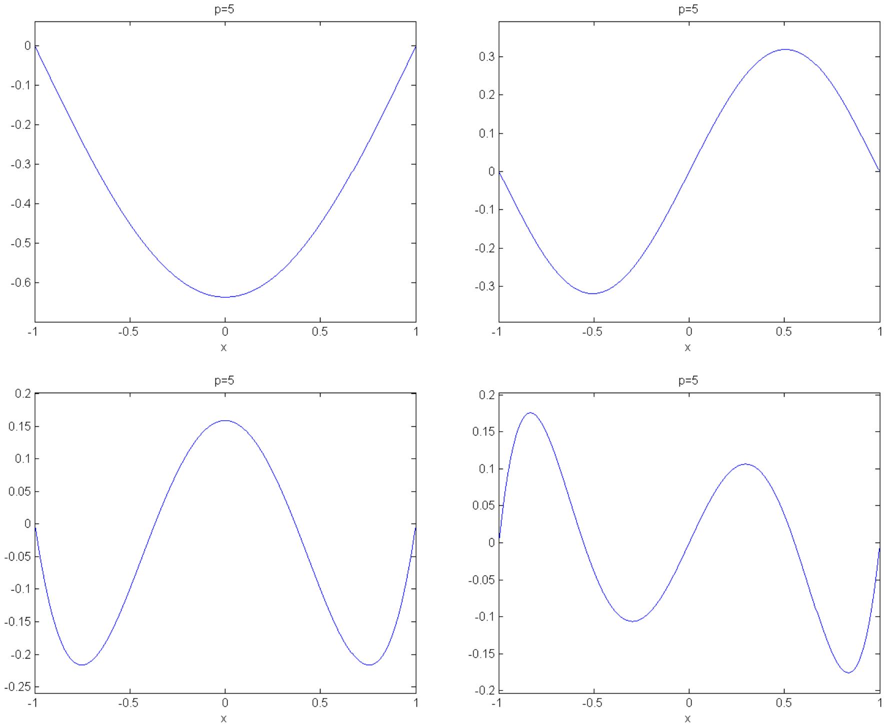

Figure 1: Generalized eigenfunctions of the Laplacian in the space $P_0^5(-1,1)$.

3 One-Dimensional Elements

In the rest of the paper we restrict ourselves to the case $L = \Delta$ (Laplace operator). First let $\hat{K} = (-1, 1)$ and $\hat{W} = P_0^p(-1, 1)$. This space has the dimension $n = p - 1$. For any $p > 1$, the n generalized eigenfunctions are computed as described in Section 2. For illustration, the generalized eigenfunctions $\psi_1, \psi_2, \ldots, \psi_4$ of polynomial degree $p = 5$ are shown in Fig. 1.

According to our numerical experiments, the generalized eigenfunctions have better conditioning properties for the *hp*-FEM discretization of the Laplace operator than any other known set of higher-order shape functions. To illustrate this fact, we compare them with the integrated Legendre polynomials (1) and with the Lagrange nodal shape functions based on Gauss-Lobatto points, which are used often in the context of spectral FEM (see, e.g., [12]).

In the left and right parts of Fig. 2 we show the condition numbers of the stiffness and mass matrices, respectively, where the Laplace operator is discretized on a single element $(-1, 1)$, using polynomial degrees $p = 2, 3, \ldots, 15$.

Results shown in Fig. 2 are in agreement with the general experience that spectral elements yield worse-conditioned stiffness matrices than *hp* finite elements but better conditioned mass matrices. Both integrated Legendre polynomials (1) and generalized eigenfunctions (7) yield an optimal stiffness matrix $S = I$. It is interesting to observe that also the mass matrices corresponding to these two sets of shape functions have (visually) identi-

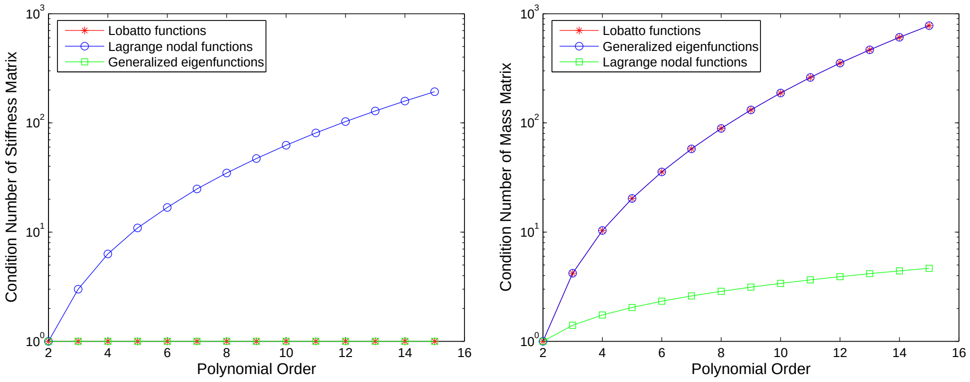

Figure 2: Conditioning of the stiffness and mass matrices, $p = 2, 3, \ldots, 15$.

cal condition numbers. This is somewhat surprising since the mass matrix corresponding to the generalized eigenfunctions is diagonal while the one of the integrated Legendre polynomials is not. After analysing the situation, it turns out that the condition numbers are exactly the same, and, moreover, that this behavior is typical whenever we deal with two bases which are orthonormal under the same inner product. Let us introduce the following result:

Lemma 3.1. *Let V be a Hilbert space with a subspace W, $\dim(V) = n < \infty$. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $\omega_1, \omega_2, \ldots, \omega_n$ be two bases in W. Further, let $a : W \times W \to \mathbb{R}$ be a symmetric bilinear form such that*

$$a(\gamma_i, \gamma_j) = a(\omega_i, \omega_j) = \delta_{ij} \tag{11}$$

for all $1 \le i, j \le n$. Then, for any other symmetric bilinear form $b : W \times W \to \mathbb{R}$, the Gram matrices $M_\gamma = \{b(\gamma_i, \gamma_j)\}_{i,j=1}^n$ and $M_\omega = \{b(\omega_i, \omega_j)\}_{i,j=1}^n$ have identical eigenvalues.

Proof. Let us express the element ω_i as

$$\omega_i = \sum_{k=1}^{n} \alpha_k^i \gamma_k, \qquad 1 \le i \le n.$$

We have

$$(M_\omega)_{ij} = b(\omega_i, \omega_j) = b\left(\sum_{k=1}^{n} \alpha_k^i \gamma_k, \sum_{m=1}^{n} \alpha_m^j \gamma_m\right) = \sum_{k=1}^{n} \sum_{m=1}^{n} \alpha_k^i \alpha_m^j b(\gamma_k, \gamma_m).$$

Hence, we can write

$$M_\omega = C_\alpha^T n_\gamma C_\alpha$$

where $C_\alpha = \{\alpha_k^i\}_{k,i=1}^n$. It follows from (11) that

$$\delta_{ij} = a(\omega_i, \omega_j) = \sum_{k=1}^{n} \sum_{m=1}^{n} \alpha_k^i \alpha_m^j \underbrace{a(\gamma_k, \gamma_m)}_{=\delta_{km}} = \sum_{k=1}^{n} \alpha_k^i \alpha_k^j.$$

Thus $C_\alpha^T = C_\alpha^{-1}$, i.e., the matrices M_γ and M_ω are similar. □

The generalized eigenfunctions become clearly distinguished from other types of higher-order shape functions on product elements of the form $(-1,1)^d$ which we study in the next section.

4 Cartesian Product Elements

Let us consider a d-dimensional cube $\hat{K} = K_d = (-1,1)^d$ equipped with a set of directional polynomial degrees $2 \le p_1, p_2, \dots, p_d$. The corresponding product space has the form

$$\hat{W} = W_d = P_0^{p_1}(-1,1) \times P_0^{p_2}(-1,1) \times \ldots \times P_0^{p_d}(-1,1).$$

The dimension of W_d is $n = \prod_{i=1}^{d}(p_i - 1)$. In this space we assume the form $a_d(\cdot,\cdot) : W_d \times W_d \to \mathbb{R}$

$$a_d(u,v) = \int_{K_d} \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x},$$

corresponding to the Laplace operator, and the L^2-product $b_d(\cdot,\cdot) : W_d \times W_d \to \mathbb{R}$,

$$b_d(u,v) = \int_{K_d} uv \, \mathrm{d}\boldsymbol{x}.$$

We construct product shape functions of the form

$$\phi_{i_1,i_2,\dots,i_d}(x_1,x_2,\dots,x_d) = \left(\frac{\prod_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}} \right)^{1/2} \prod_{m=1}^{d} \psi_{i_m}^{(p_m)}(x_m), \; 1 \le i_m \le p_m - 1, \tag{12}$$

where $\psi_1^{(p_m)}$, $\psi_2^{(p_m)}$, $\dots$, $\psi_{p_m-1}^{(p_m)}$ are the one-dimensional generalized eigenfunctions of degree p_m from Section 3 and $\lambda_1^{(p_m)}$, $\lambda_2^{(p_m)}$, $\dots$, $\lambda_{p_m-1}^{(p_m)}$ are the corresponding eigenvalues. It is easy to see that polynomials (12) are linearly independent and that they span the space W_d. We have the following lemma:

Lemma 4.1. *Shape functions (12) satisfy*

$$a_d(\phi_{i_i,i_2,\dots,i_d}, v) = \left(\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)} \right) b_d(\phi_{i_i,i_2,\dots,i_d}, v) \qquad \text{for all } v \in W_d, \tag{13}$$

i.e., they are the generalized eigenfunctions of the Laplacian in the space W_d *with the eigenvalues* $\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}$*. Moreover, we have analogies to (9) and (10),*

$$a_d(\phi_{i_1,i_2,\dots,i_d}, \phi_{j_1,j_2,\dots,j_d}) = \prod_{m=1}^{d} \delta_{i_m j_m} \tag{14}$$

and

$$b_d(\phi_{i_1,i_2,\dots,i_d}, \phi_{j_1,j_2,\dots,j_d}) = \frac{\prod_{m=1}^{d} \delta_{i_m j_m}}{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}. \tag{15}$$

Proof. It is sufficient to prove (13) for an arbitrary basis of W_d, which means that it is enough to prove (14) and (15). For an arbitrary set of directional polynomial degrees $2 \leq p_1, p_2, \ldots, p_d$ we proceed by induction over the spatial dimension d. We know from Lemma 2.2 that (14) and (15) hold for $d = 1$. Let $\boldsymbol{x}_d = (x_1, x_2, \ldots, x_d)$. It follows from (12) that

$$\phi_{i_1,i_2,\ldots,i_{d+1}}(\boldsymbol{x}_{d+1}) = \psi_{i_{d+1}}^{(p_{d+1})}(x_{d+1}) \left(\lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \right)^{1/2} \phi_{i_1,i_2,\ldots,i_d}(\boldsymbol{x}_d).$$

We have

$$\begin{aligned} & a_{d+1}(\phi_{i_1,i_2,\ldots,i_{d+1}}, \phi_{j_1,j_2,\ldots,j_{d+1}}) \\ &= b(\psi_{i_{d+1}}^{(p_{d+1})}, \psi_{j_{d+1}}^{(p_{d+1})}) \left(\lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \right)^{\frac{1}{2}} \\ &\cdot \left(\lambda_{j_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{j_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{j_m}^{(p_m)}} \right)^{\frac{1}{2}} a_d(\phi_{i_1,i_2,\ldots,i_d}, \phi_{j_1,j_2,\ldots,j_d}) \\ &+ a(\psi_{i_{d+1}}^{(p_{d+1})}, \psi_{j_{d+1}}^{(p_{d+1})}) \left(\lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \right)^{\frac{1}{2}} \\ &\cdot \left(\lambda_{j_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{j_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{j_m}^{(p_m)}} \right)^{\frac{1}{2}} b_d(\phi_{i_1,i_2,\ldots,i_d}, \phi_{j_1,j_2,\ldots,j_d}). \end{aligned}$$

Using the induction assumptions (14) and (15), this yields

$$\begin{aligned} a_{d+1}(\phi_{i_1,i_2,\ldots,i_{d+1}}, \phi_{j_1,j_2,\ldots,j_{d+1}}) &= \frac{\delta_{i_{d+1}j_{d+1}}}{\lambda_{i_{d+1}}^{(p_{d+1})}} \lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \prod_{m=1}^{d} \delta_{i_m j_m} \\ + \delta_{i_{d+1}j_{d+1}} \lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \frac{\prod_{m=1}^{d} \delta_{i_m j_m}}{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}} &= \prod_{m=1}^{d+1} \delta_{i_m j_m}. \end{aligned}$$

For the form b_{d+1} we obtain

$$\begin{aligned} & b_{d+1}(\phi_{i_1,i_2,\ldots,i_{d+1}}, \phi_{j_1,j_2,\ldots,j_{d+1}}) \\ &= b(\psi_{i_{d+1}}^{(p_{d+1})}, \psi_{j_{d+1}}^{(p_{d+1})}) \left(\lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \right)^{\frac{1}{2}} \\ &\cdot \left(\lambda_{j_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{j_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{j_m}^{(p_m)}} \right)^{\frac{1}{2}} b_d(\phi_{i_1,i_2,\ldots,i_d}, \phi_{j_1,j_2,\ldots,j_d}) \\ &= \frac{\delta_{i_{d+1}j_{d+1}}}{\lambda_{i_{d+1}}^{(p_{d+1})}} \lambda_{i_{d+1}}^{(p_{d+1})} \frac{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \frac{\prod_{m=1}^{d} \delta_{i_m j_m}}{\sum_{m=1}^{d} \lambda_{i_m}^{(p_m)}} = \frac{\prod_{m=1}^{d+1} \delta_{i_m j_m}}{\sum_{m=1}^{d+1} \lambda_{i_m}^{(p_m)}} \end{aligned}$$

which finishes the proof. □

Fig. 3 presents a comparison of the conditioning properties of generalized eigenfunctions (12) to two other frequently used sets of higher-order shape functions for quadrilateral elements: to Lagrange nodal shape functions based on product Gauss-Lobatto points [21] and to products of integrated Legendre polynomials (1). Fig. 3 shows the condition numbers of stiffness and mass matrices in the space W_2 for $2 \leq p_1 = p_2 \leq 9$.

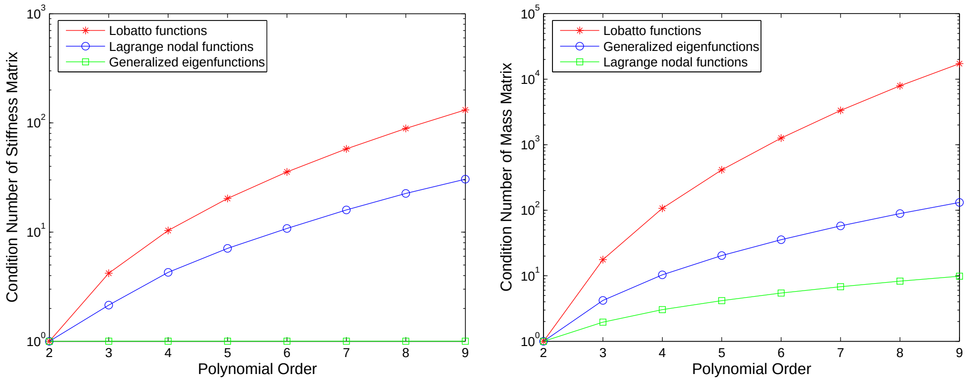

Figure 3: Conditioning of the stiffness and mass matrices for $p_1 = p_2 = 2, 3, \ldots, 9$.

In contrast to the one-dimensional case shown in Fig. 2, we can see in the left part of Fig. 3 that the generalized eigenfunctions (12) yield best-conditioned stiffness matrices. As in the one-dimensional case, the Lagrange product shape functions yield mass matrices with lowest condition numbers. Let us stress that mass matrices obtained with the generalized eigenfunctions (12) are diagonal.

5 Simplicial Elements

On a simplex $\hat{K} \subset \mathbb{R}^d$ we only have one polynomial degree $p \geq d+1$ and the corresponding polynomial space has the form

$$\hat{W} = P_0^p(\hat{K}) = \{u \in P^p(\hat{K});\ u = 0 \text{ on } \partial\hat{K}\}. \tag{16}$$

The dimension of $\hat{W}$ is

$$n = \frac{\prod_{k=1}^{d}(p-k)}{d!}.$$

Generalized eigenfunctions of the Laplacian in the space $\hat{W}$ are computed following the procedure described in Section 2.

For illustration, we consider a reference triangle $\hat{K} = K_t$ with the vertices $[-1,-1], [1,-1], [-1,1]$. The corresponding Matlab script and a text data file containing the coefficients y_{jk} for polynomial degrees $3 \leq p \leq 10$ can be downloaded from the web page `http://hpfem.math.utep.edu`. In Fig. 4 we compare the conditioning properties of the generalized eigenfunctions to the Lagrange nodal shape functions based on Fekete points [26] (best nodal shape functions available for triangular elements) and to shape functions based on integrated Legendre polynomials [22] (most widely used set of higher-order shape functions for continuous *hp* elements on triangular meshes).

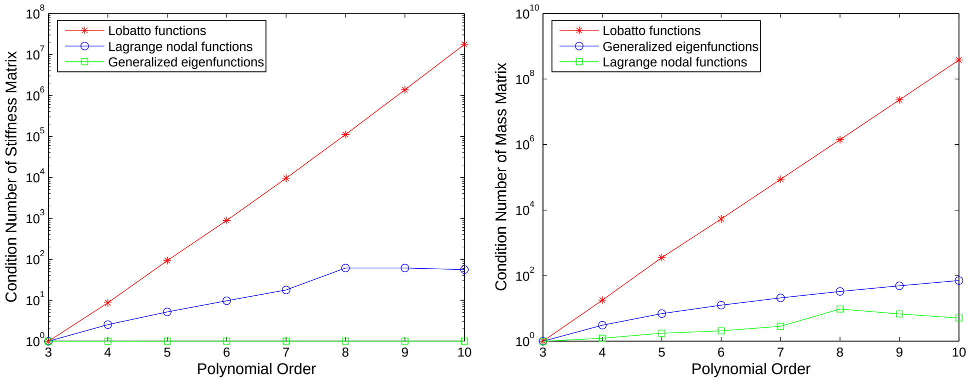

Figure 4: Conditioning of the stiffness and mass matrices for $p = 3, 4, \ldots, 10$.

We can see in Fig. 4 that the shape functions based on integrated Legendre polynomials exhibit a particularly poor exponential growth of both condition numbers in this case. Again, the generalized eigenfunctions yield optimally-conditioned stiffness matrices and the Lagrange nodal shape functions yield best-conditioned mass matrices. The somewhat strange behavior of the conditioning curves of the Lagrange nodal shape functions for $p = 8, 9$ was consulted with the authors of the Fekete points [26] who reported the same experience.

The following remark applies to all domains $\hat{K}$:

Remark 5.1. *Let an eigenfunction-eigenvalue pair* (φ, λ) *solve the generalized eigenproblem for the Laplacian (5). Then for any positive* $k \in \mathbb{R}$*, the eigenfunction-eigenvalue pair* $(\varphi, \lambda + k)$ *solves the generalized eigenproblem for the operator* $Lu = -\Delta u + ku$*,*

$$\int_{\hat{K}} \nabla\varphi \cdot \nabla v + k\varphi v \, d\boldsymbol{x} = (\lambda + k) \int_{\hat{K}} \varphi v \, d\boldsymbol{x} \qquad \text{for all } v \in \hat{W}. \tag{17}$$

6 Role of Reference Maps

Most *hp*-FEM and spectral (SFEM) codes use the so-called *affine concept* of FEM. This is an economical approach where one defines shape functions and quadrature rules on a suitable reference domain $\hat{K}$ and maps all physical mesh elements K_i onto $\hat{K}$ by means of bijective reference maps $\boldsymbol{x}_{K_i} : \hat{K} \to K_i$. Substitution theorem is then used to transform all integrals in the weak formulation from mesh elements onto $\hat{K}$, and all contributions to the stiffness/mass matrices are calculated on $\hat{K}$ only.

Every mesh triangle K_i, however, can be mapped onto $\hat{K}$ using six different affine maps. This number reduces to three when $\hat{K}$ and K_i have the same orientation of vertices, as shown in Fig. 5.

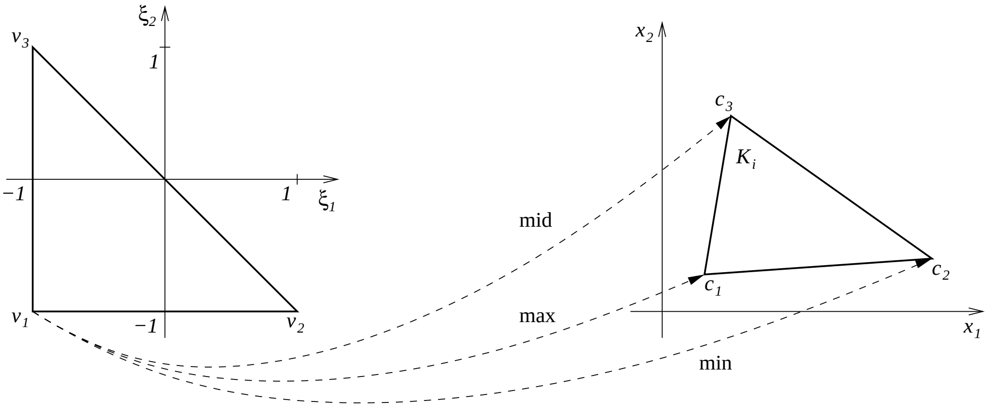

Figure 5: Three different ways to map a physical mesh triangle K_i onto a reference triangle.

In Fig. 5, the central vertex v_1 of $\hat{K}$ is mapped onto the vertex with maximum, medium, and minimum angle in K_i. It was shown in [24] that the selection of reference maps influences the condition number of the discrete problem significantly. This dependence varies with the choice of higher-order shape functions and geometry of the mesh.

Let us use the standard L-shape domain benchmark problem [22] with singular solution to illustrate this effect. Solved is the Laplace equation $-\Delta u = 0$ in the domain $\Omega = (-1,1)^2 \setminus [-1,0]^2$, and the Dirichlet boundary conditions are chosen to match the exact solution u,

$$u(\boldsymbol{x}) = R(\boldsymbol{x})^{2/3} \sin(2\theta(\boldsymbol{x})/3 + \pi/3).$$

Here, $R(\boldsymbol{x})$ and $\theta(\boldsymbol{x})$ are the standard spherical coordinates in $\mathbb{R}^2$. The domain Ω and the geometry of the mesh are shown in Fig. 6. The solution u and the magnitude of its gradient $|\nabla u|$ (truncated for visualization purposes) are depicted in Fig. 7.

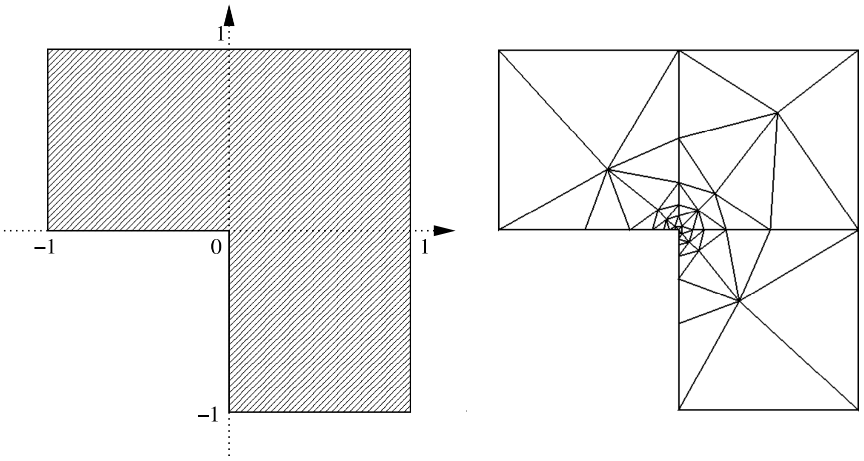

Figure 6: The L-shape domain and its partition.

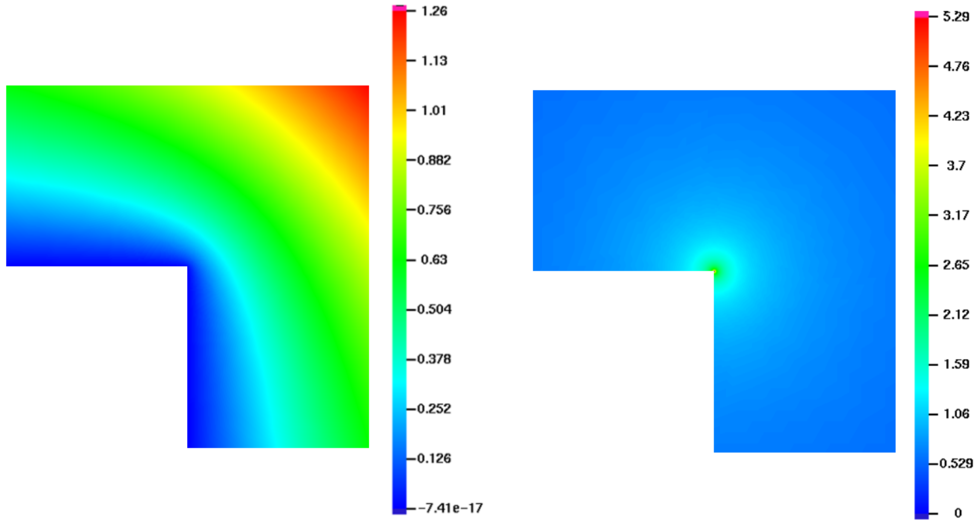

Figure 7: Exact solution u (left) and the norm of its gradient (right).

The following Figs. 8, 9, and 10 show the dependence of the condition number of the stiffness matrix on the choice of the reference maps for the standard shape functions based on integrated Legendre polynomials [1, 22], more recent shape functions based on integrated Jacobi polynomials [8], and the shape functions based on generalized eigenfunctions of the Laplace operator, respectively. The horizontal axis represents a uniform polynomial degree of mesh elements. The vertical axis shows the condition number of the stiffness matrices obtained with the max. angle, mid. angle, and min. angle maps (same for all elements). For better illustration, the values of the condition numbers are made relative to the condition number obtained when the central vertex v_1 of $\hat{K}$ is mapped onto the vertex in K_i with the lowest index for all elements.

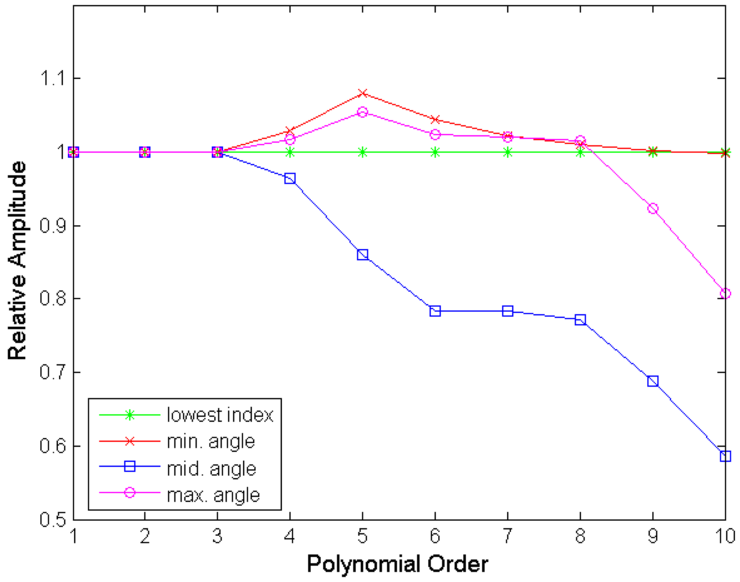

Figure 8: Condition number of stiffness matrices, shape functions based on integrated Legendre polynomials, $p = 1, 2, \ldots, 10$.

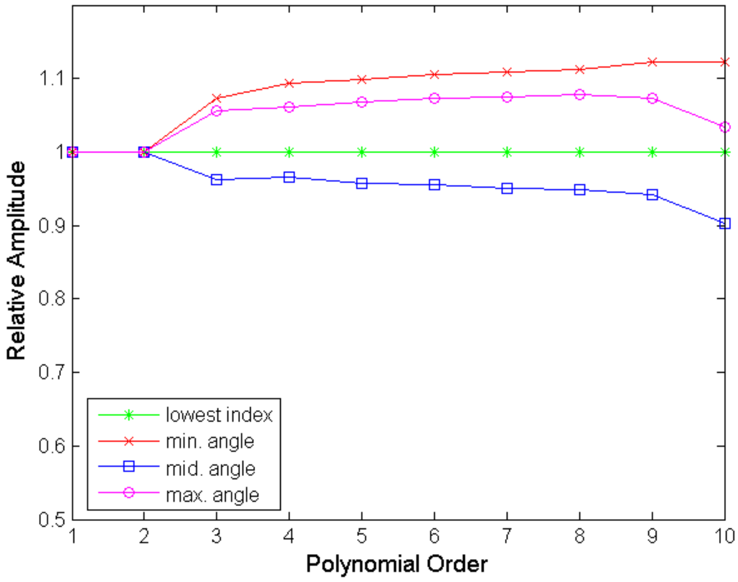

Figure 9: Condition number of stiffness matrices, shape functions based on integrated Jacobi polynomials, $p = 1, 2, \ldots, 10$.

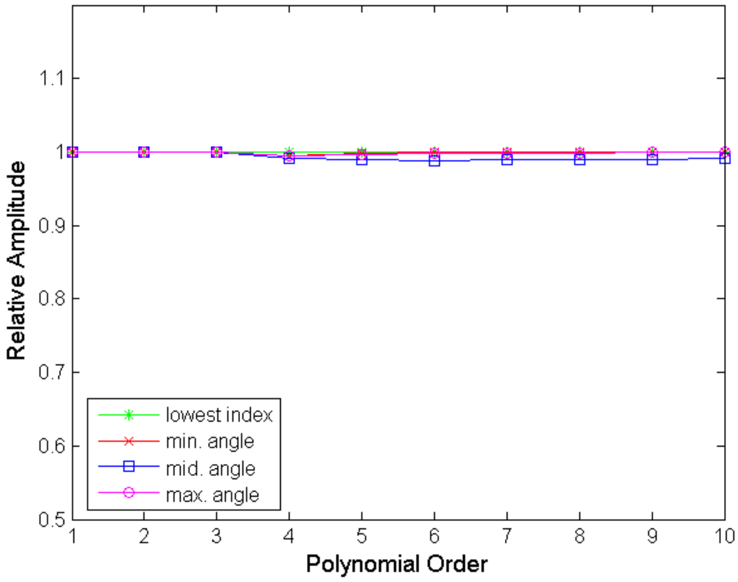

Figure 10: Condition number of stiffness matrices, shape functions based on generalized eigenfunctions of the Laplacian, $p = 1, 2, \ldots, 10$.

Figs. 8 – 10 correspond to the L-shape domain problem, but they represent a general trend that we observed in various other computations: The influence of the reference maps on the discrete problem is strongest for the standard shape functions based on integrated Legendre polynomials, slightly weaker for the shape functions based on integrated Jacobi polynomials, and it almost vanishes for the shape functions based on generalized eigenfunctions of the Laplace operator.

In addition, Fig. 11 shows the condition number of the stiffness matrices for the three above mentioned sets of shape functions and polynomial degrees $p = 1, 2, \ldots, 10$. For every case we use the reference map which yields the minimum condition number.

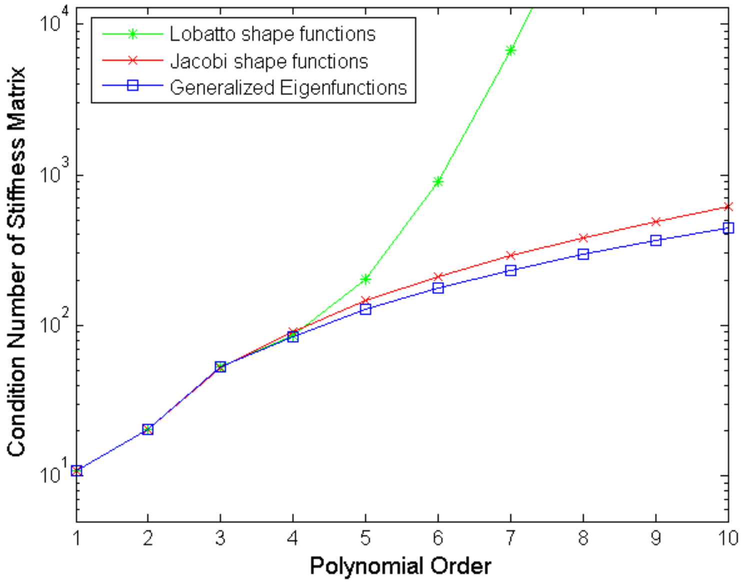

Figure 11: Condition number of stiffness matrices, $p = 1, 2, \ldots, 10$.

Again, the results shown in Fig. 11 correspond to the L-shape domain problem, but at the same time they represent a general trend that we observed in various other computations: The shape functions based on integrated Legendre polynomials yield quite ill-conditioned matrices compared to the other two sets of shape functions, and the gap opens very fast as the polynomial degree in elements is increased. Conditioning properties of the shape functions based on the integrated Jacobi polynomials and generalized eigenfunctions is similar, with the latter set being slightly better in most computations.

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