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Abstract: *We present a proof of the discrete maximum principle (DMP) for the Poisson equation* $-\Delta u = f$ *equipped with mixed Dirichlet-Neumann boundary conditions. The problem is discretized using finite elements of arbitrary lengths and polynomial degrees (hp-FEM). In contrast to the Dirichlet case, with mixed Dirichlet-Neumann boundary conditions the DMP holds without any limitations on the mesh or polynomial degrees of the elements.*

AMS subject classification: 65N30, 35B50

Keywords: Discrete maximum principle, higher order elements, *hp*-FEM, Poisson equation, mixed boundary conditions.

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1 Introduction

It is well known that finite element solutions to elliptic and parabolic PDEs sometimes exhibit behavior which is incompatible with the corresponding maximum principles (and, consequently, incompatible with the underlying physics). Most frequently this happens when a finite element mesh contains large dihedral angles, but also in other situations. Discrete maximum principles (DMP) provide additional restrictions on finite element meshes under which the maximum principles are preserved on the discrete level.

The first DMP were introduced in the 1970s and used to prove the convergence of finite differences and lowest-order finite element methods (see, e.g., [3, 4]). Nowadays the DMP play an important role in computational PDEs by guaranteeing that approximation of physically nonnegative quantities such as the density, temperature, concentration, or electric charge remains nonnegative. Due to the difficulty of the topic, current research in the area of DMP almost exclusively deals with lowest-order elements (see, e.g., [9, 10, 11, 19, 20]). However, in the last decades, significant progress has been made in the development of the *hp*-FEM (finite element methods with variable size and polynomial degree of elements) and their applications to challenging large-scale problems in computational science and engineering (see, e.g., [1, 13, 14, 17]). These methods are substantially more efficient compared to standard lowest-order schemes, and an increasing demand for them implies a need for the corresponding generalizations of the DMP.

It was shown in [16] that the DMP cannot be extended from lowest-order FEM to *hp*-FEM in a straightforward manner, and a weak DMP was introduced. Recently, a maximum principle for one-dimensional Poisson equation equipped with Dirichlet boundary conditions and discretized by *hp*-FEM was presented in [21]. The result was proved under a mild sufficient condition stating that the length of the longest element in the mesh must be less than 90% of the length of the entire domain. In this paper we generalize this result to the mixed Dirichlet-Neumann boundary conditions. The extension is not straightforward since the inverse stiffness matrix has a different structure.

2 The model problem and its discretization

We solve the one dimensional Poisson equation with mixed Dirichlet-Neumann boundary conditions,

$$\begin{aligned} -u'' &= f \quad \text{in } \Omega, \\ u(\alpha) &= 0, \\ u'(\beta) &= g(\beta). \end{aligned}$$

Here, $\Omega = (\alpha, \beta) \subset \mathbb{R}$ is an interval.

The corresponding weak formulation reads: Find $u \in V$ such that

$$a(u, v) = (f, v) + g(\beta)v(\beta) \quad \forall v \in V, \tag{1}$$

where $V = \{v \in H^1(\Omega) : v(\alpha) = 0\}$, $f \in L^2(\Omega)$ is a right-hand side, $g(\beta) \in \mathbb{R}$, $(\cdot, \cdot)$ stands for an $L^2(\Omega)$ inner product, and $a(u, v) = (u', v')$.

In a standard way we create a partition $\alpha = x_0 < x_1 < \ldots < x_M = \beta$ of the domain Ω consisting of M elements $K_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, M$. Every element K_i is assigned an arbitrary polynomial degree $p_i \geq 1$. The corresponding finite element space of piecewise-polynomial continuous functions $V_{hp} \subset V$ has the form

$$V_{hp} = \{v_{hp} \in V;\ v_{hp}|_{K_i} \in P^{p_i}(K_i),\ i = 1, 2, \ldots, M\}.$$

Here $P^{p_i}(K_i)$ stands for the space of polynomials of degree at most p_i on the element K_i. The space V_{hp} has the dimension $N = \sum_{i=1}^{M} p_i$. There exists a unique function $u_{hp} \in V_{hp}$ satisfying

$$a(u_{hp}, v_{hp}) = (f, v_{hp}) + g(\beta)v(\beta) \quad \forall v_{hp} \in V_{hp}. \tag{2}$$

Definition 2.1. *Problem* (2) *satisfies the* discrete maximum principle *(DMP) if*

$$f \leq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \leq 0 \quad \Rightarrow \quad \max_{\overline{\Omega}} u_{hp} = \max_{\partial\Omega} u_{hp},$$

where $\partial\Omega$ is the boundary of the domain Ω.

Definition 2.2. *Problem* (2) *satisfies the* discrete minimum principle *if*

$$f \geq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \geq 0 \quad \Rightarrow \quad \min_{\overline{\Omega}} u_{hp} = \min_{\partial\Omega} u_{hp}.$$

Definition 2.3. *Problem* (2) conserves nonnegativity *if*

$$f \geq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \geq 0 \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } \Omega.$$

Clearly, the discrete maximum and minimum principles are equivalent for problem (2). We will use the following lemma to prove the DMP via conservation of nonnegativity.

Lemma 2.1. *If problem* (2) *conserves nonnegativity then it satisfies the discrete maximum principle.*

Proof. Since $u_{hp} \geq 0$ in Ω and $u_{hp}(\alpha) = 0$, we conclude

$$\min_{\partial\Omega} u_{hp} = 0 = \min_{\overline{\Omega}} u_{hp}.$$

□

Remark 2.1. *For the sake of simplicity, we formulated problem* (2) *with a homogeneous Dirichlet boundary condition* $u(\alpha) = 0$*. However, all results of this study hold for a nonhomogeneous condition of the form* $u(\alpha) = u_\alpha$*. Namely, every solution* $\hat{u}_{hp}$ *to problem* (2) *with nonhomogeneous condition* $u(\alpha) = u_\alpha$ *can be decomposed to* $\hat{u}_{hp} = u^C_{hp} + u_{hp}$*, where* u^C_{hp} *is a constant function such that* $u^C_{hp}(\alpha) = u_\alpha$ *and* u_{hp} *vanishes at the endpoint* α*.*

3 Discrete Green's Function

The discrete Green's function (DGF) is defined in analogy to the standard Green's function:

Definition 3.1. *For an arbitrary* $z \in \Omega$*, the unique solution* $G_{hp,z} \in V_{hp}$ *to the problem*

$$a(v_{hp}, G_{hp,z}) = v_{hp}(z) \quad \forall v_{hp} \in V_{hp} \tag{3}$$

is called the discrete Green's function *(DGF) corresponding to the point* z*.*

In the following, we will use the notation $G_{hp}(x, z) = G_{hp,z}(x)$. A combination of (2) and (3) yields the so-called Kirchhoff-Helmholtz representation

$$u_{hp}(z) = \int_\Omega G_{hp}(x, z) f(x)\, \mathrm{d}x + g(\beta) G_{hp}(\beta, z) \quad \forall z \in \Omega. \tag{4}$$

The following lemma shows that the DGF can easily be expressed using any basis of V_{hp}, cf. [5]. We use the Kronecker symbol

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases}$$

Lemma 3.1. *Let* $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ *be basis of* V_{hp}*. If the stiffness matrix* $A_{ij} = a(\varphi_j, \varphi_i)$*,* $1 \leq i, j \leq N$ *is nonsingular, then*

$$G_{hp}(x, z) = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{-1}_{jk} \varphi_k(x) \varphi_j(z). \tag{5}$$

Here, A^{-1}_{jk} *are the entries of the inverse stiffness matrix, i.e.,* $\sum_{j=1}^{N} A_{ij} A^{-1}_{jk} = \delta_{ik}$*,* $1 \leq i, k \leq N$*.*

Proof. Substitute

$$G_{hp}(x, z) = \sum_{i=1}^{N} c_i(z) \varphi_i(x) \tag{6}$$

into (3) with $v_{hp} = \varphi_j$. It follows that

$$\sum_{i=1}^{N} c_i(z) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(z).$$

The coefficients $c_i(z)$ are expressed in terms of the inverse matrix as $c_k(z) = \sum_{j=1}^{N} \varphi_j(z) A^{-1}_{jk}$, and they are substituted back into (6). This finishes the proof. □

Theorem 3.1. *Problem* (2) *conserves nonnegativity if and only if the corresponding discrete Green's function* $G_{hp}(x, z) = G_{hp,z}(x)$ *defined by* (3) *is nonnegative in* Ω^2.

Proof. By (5), the discrete Green's function $G_{hp}(x, z)$ is continuous up to the boundary of Ω. The rest follows immediately from (4). □

This theorem is a useful tool for the analysis of discrete maximum principles. In the rest of this paper we will show that the discrete Green's function corresponding to the problem (2) is nonnegative.

4 DGF for the model problem

4.1 Lowest-Order Case

In this section we will construct the DGF for problem (2). We begin with the case $p_1 = p_2 = \ldots = p_M = 1$. Let us define $h_i = x_i - x_{i-1}$. By $\mathcal{B}^L = \{\phi_1, \phi_2, \ldots, \phi_M\}$ we denote the standard lowest-order basis consisting of the piecewise-linear "hat functions" such that $\phi_j(x_i) = \delta_{ij}$, $1 \le i, j \le M$. In this case the stiffness matrix $A^L \in \mathbb{R}^{M\times M}$ is tridiagonal,

$$A^L_{ij} = \begin{cases} 1/h_i + 1/h_{i+1} & \text{for } i = j < M, \\ 1/h_M & \text{for } i = j = M, \\ -1/h_{i+1} & \text{for } i = j - 1, \\ -1/h_{i-1} & \text{for } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j = 1, 2, \ldots, M$.

Lemma 4.1. *The inverse matrix* $(A^L)^{-1} \in \mathbb{R}^{M\times M}$ *has the form*

$$(A^L)^{-1} = \begin{pmatrix} x_1 - \alpha & x_1 - \alpha & x_1 - \alpha & \ldots & x_1 - \alpha \\ x_1 - \alpha & x_2 - \alpha & x_2 - \alpha & \ldots & x_2 - \alpha \\ x_1 - \alpha & x_2 - \alpha & x_3 - \alpha & \ldots & x_3 - \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 - \alpha & x_2 - \alpha & x_3 - \alpha & \ldots & x_M - \alpha \end{pmatrix},$$

i.e., $(A^L)^{-1}_{ij} = x_i - \alpha$ *for* $1 \le i \le j \le M$ *and* $(A^L)^{-1}_{ij} = x_j - \alpha$ *for* $1 \le j < i \le M$.

Proof. We want to show that $z_{ij} = \delta_{ij}$, where

$$z_{ij} = \sum_{k=1}^{M} (A^L)^{-1}_{ik} A^L_{kj} = \sum_{k=1}^{M} (A^L)^{-1}_{ik} a(\phi_j, \phi_k),$$

for all $i, j = 1, 2, \ldots, M$. We fix i and j, and consider the bilinear forms

$$a_1(u, v) = \int_{\alpha}^{x_i} u'v' \,\mathrm{d}x \quad \text{and} \quad a_2(u, v) = \int_{x_i}^{\beta} u'v' \,\mathrm{d}x.$$

We use the explicit formulae for $(A^L)^{-1}_{ik}$ to get

$$z_{ij} = a\Big(\phi_j, \sum_{k=1}^{i-1} (x_k - \alpha)\phi_k\Big) + (x_i - \alpha) a(\phi_j, \phi_i) + (x_i - \alpha) a\Big(\phi_j, \sum_{k=i+1}^{M} \phi_k\Big).$$

Now, we split the term $a(\phi_j, \phi_i) = a_1(\phi_j, \phi_i) + a_2(\phi_j, \phi_i)$ to obtain

$$z_{ij} = a_1(\phi_j, x - \alpha) + (x_i - \alpha) a_2(\phi_j, 1) = a_1(\phi_j, x - \alpha) = \delta_{ij},$$

where the last equality follows from a straightforward simple computation. □

Using Proposition 4.1 and identity (5), we can write the DGF in the form

$$\begin{aligned} G^L_{hp}(x, z) = &\sum_{i=1}^{M} (x_i - \alpha)\phi_i(x)\phi_i(z) \\ &+ \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} (x_i - \alpha)[\phi_i(x)\phi_j(z) + \phi_j(x)\phi_i(z)]. \end{aligned} \tag{7}$$

In particular, we see immediately that

$$G^L_{hp}(x, z) \ge 0 \quad \forall [x, z] \in \Omega^2. \tag{8}$$

The situation is illustrated in Figure 1.

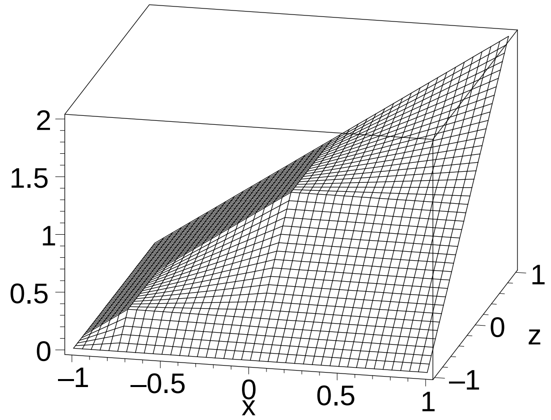

Figure 1: The lowest-order part $G^L_{hp}(x,z)$ of the discrete Green's function $G_{hp}(x,z)$ for the Poisson equation with mixed boundary conditions in $\Omega = (-1,1)$ on a mesh with three elements $[-1,-3/4]$, $[-3/4,0]$, and $[0,1]$.

4.2 Higher-Order Case

In this paragraph we return to the original setting with arbitrary polynomial degrees $p_i \ge 1$. In order to facilitate the construction of higher-order basis functions of the space V_{hp}, let us introduce the Lobatto shape functions $l_0, l_1, l_2, \ldots$ on a reference interval $\hat{K} = [-1,1]$ (see, e.g., [14, 17]).

The lowest-order Lobatto shape functions l_0 and l_1 have the form $l_0(\xi) = (1-\xi)/2$, $l_1(\xi) = (1+\xi)/2$, $\xi \in \hat{K}$. The higher-order shape functions $l_2, l_3, \ldots$ are defined as antiderivatives to the Legendre polynomials. Therefore, they satisfy

$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}, \quad i,j = 2,3,\ldots.$$

Every Lobatto shape function l_i, $i = 2,3,\ldots$, is a polynomial of degree i and it vanishes at ± 1. Thus it can be expressed as

$$l_i(\xi) = l_0(\xi) l_1(\xi) \kappa_i(\xi), \quad i = 2,3,\ldots,$$

where κ_i is a polynomial of degree $i-2$. For reference, a first few kernels κ_i are listed in Appendix.

The basis $\mathcal{B} = \{\phi_1, \phi_2, \ldots, \phi_N\}$ of V_{hp} can be written as $\mathcal{B} = \mathcal{B}^L \cup \mathcal{B}^B$, where $\mathcal{B}^L$ was defined above and $\mathcal{B}^B$ is the higher-order part of the basis comprising functions $\phi_M, \phi_{M+1}, \ldots, \phi_N$. These are defined as follows:

Consider the standard affine transformations from $\hat{K}$ to K_i,

$$\chi_{K_i}(\xi) = \frac{(x_i - x_{i-1})\xi + (x_i + x_{i-1})}{2}. \tag{9}$$

On an element K_i of the polynomial degree p_i, there are $p_i - 1$ higher-order basis functions. These vanish outside of K_i and in K_i they are defined as the Lobatto shape functions $l_2, l_3, \ldots, l_{p_i}$ composed with the inverse map $\chi_{K_i}^{-1}(x)$.

Proposition 4.1. *We have the following orthogonality relations:*

$$a(\phi^L, \phi^B) = 0 \quad \forall \phi^L \in \mathcal{B}^L,\ \forall \phi^B \in \mathcal{B}^B,$$
$$a(\phi^B, \psi^B) = 0 \quad \forall \phi^B \in \mathcal{B}^B,\ \forall \psi^B \in \mathcal{B}^B,\ \phi^B \neq \psi^B.$$

Proof. The proof is straightforward, based on the L^2-orthogonality of the Legendre polynomials. □

By Proposition 4.1, both the stiffness matrix A and its inverse have the following block structure:

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

with

$$D = \mathrm{diag}\Big(\underbrace{\frac{2}{h_1}, \ldots, \frac{2}{h_1}}_{(p_1-1) \text{ times}}, \underbrace{\frac{2}{h_2}, \ldots, \frac{2}{h_2}}_{(p_2-1) \text{ times}}, \ \ldots\ , \underbrace{\frac{2}{h_M}, \ldots, \frac{2}{h_M}}_{(p_M-1) \text{ times}}\Big). \tag{10}$$

By (5), the DGF can be written as

$$G_{hp}(x, z) = G^L_{hp}(x, z) + G^B_{hp}(x, z), \tag{11}$$

where $G^L_{hp}(x, z)$ corresponds to (7) and

$$G^B_{hp}(x, z) = \sum_{k=M}^{N} D^{-1}_{kk} \phi_k(x) \phi_k(z) \quad \forall [x, z] \in \Omega^2. \tag{12}$$

Unfortunately, $G^B_{hp}(x, z)$ defined by (12) is not nonnegative in the entire Ω^2 in general. For instance, in the example shown in Figure 2, there are small regions near the points $[1, 0]$ and $[0, 1]$, where the function $G^B_{hp}(x, z)$ is negative.

Notice that any partition of Ω produces a rectangular grid on Ω^2, and that $G^B_{hp}(x, z)$ can be nonzero within the diagonal squares of this grid only. In other words,

$$\mathrm{supp}\, G^B_{hp} \subset \bigcup_{i=1}^{M} K_i^2. \tag{13}$$

Lemma 4.2. *The discrete Green's function G_{hp} defined by* (11) *is nonnegative in $\Omega^2 \setminus \bigcup_{i=1}^{M} K_i^2$.*

Proof. Consider (13) together with (8). □

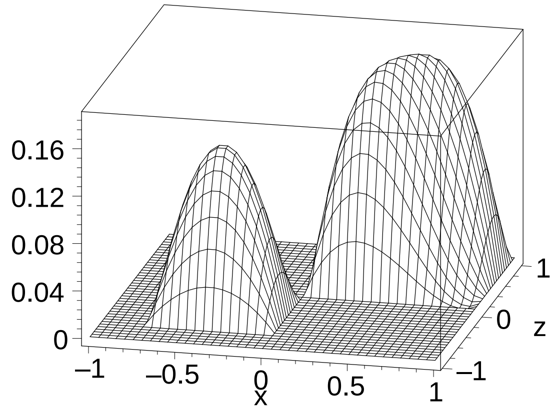

Figure 2: The higher-order part $G_{hp}^{B}(x,z)$ of the discrete Green's function $G_{hp}(x,z)$ for the Poisson equation with mixed boundary conditions in $\Omega = (-1,1)$, on a mesh with three elements $[-1,-3/4]$, $[-3/4,0]$, and $[0,1]$ of the polynomial degrees $p_1 = 1$, $p_2 = 2$, $p_3 = 3$.

5 The DGF on K_i^2

As justified by Lemma 4.2, we only need to continue with the study of the discrete Green's function $G_{hp}(x,z)$ in the union of the diagonal squares $\bigcup_{i=1}^{M} K_i^2$. Without loss of generality, let us restrict ourselves to only one square K_i^2, $1 \le i \le M$. Let $p = p_i$ be the polynomial degree assigned to K_i. Notice that only a few terms in (7) and (12) are nonzero in K_i^2. Hence, by (7), (10), and (12) we obtain

$$\begin{aligned} G_{hp}(x,z)\big|_{K_i^2} &= (x_i-\alpha)\phi_i(x)\phi_i(z) + (x_{i-1}-\alpha)\phi_{i-1}(x)\phi_{i-1}(z) \\ &\quad + (x_{i-1}-\alpha)\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right] \\ &\quad + \frac{x_i - x_{i-1}}{2} G_{hp}^{B}(x,z), \end{aligned} \tag{14}$$

$[x,z] \in K_i^2$, $1 \le i \le M$. It is convenient to introduce the notation $K_i = [x_{i-1}, x_i] = [L, R]$.

We transform the function G_{hp} from K_i^2 to the reference square $\hat{K}^2 = [-1,1]^2$ using the linear transformation (9) with $x = \chi_{K_i}(\xi)$ and $z = \chi_{K_i}(\eta)$,

$$\begin{aligned} G_{hp}(x,z)\big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) &= (R-\alpha)l_1(\xi)l_1(\eta) + (L-\alpha)l_0(\xi)l_0(\eta) \\ &\quad + (L-\alpha)\left[l_1(\xi)l_0(\eta) + l_0(\xi)l_1(\eta)\right] \\ &\quad + \frac{R-L}{2}\hat{G}_{hp}^{p,B}(\xi,\eta), \end{aligned} \tag{15}$$

$[\xi,\eta] \in \hat{K}^2$. Here $l_0(\xi)$ and $l_1(\xi)$ are the above-defined lowest-order shape functions on $\hat{K}$ and

$$\hat{G}_{hp}^{p,B}(\xi,\eta) = \sum_{k=2}^{p} l_k(\xi)l_k(\eta) = l_0(\xi)l_0(\eta)l_1(\xi)l_1(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta) \tag{16}$$

is the higher-order part.

Let us modify formula (15) in the following way: Divide (15) by $R-L>0$ and use the identities

$$\frac{R-\alpha}{R-L} = \frac{L-\alpha}{R-L} + 1,$$

and

$$l_0(\xi)l_0(\eta) + l_1(\xi)l_1(\eta) + l_0(\xi)l_1(\eta) + l_1(\xi)l_0(\eta) = 1 \quad \forall[\xi,\eta] \in \hat{K}^2.$$

We obtain

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{L-\alpha}{R-L} + l_1(\xi)l_1(\eta) + \frac{1}{2}\hat{G}_{hp}^{p,B}(\xi,\eta). \tag{17}$$

Using (16), this formula can be reshaped into

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{L-\alpha}{R-L} + l_1(\xi)l_1(\eta)\left[1 + \frac{1}{2}l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)\right]. \tag{18}$$

Clearly, $(L-\alpha)/(R-L) \geq 0$ and $l_1(\xi)l_1(\eta) \geq 0$ in $\hat{K}^2$. In [21] it was verified that the expression in the square brackets is nonnegative for $[\xi,\eta] \in \hat{K}$ and $p \leq 100$. More precisely, for $2 \leq p \leq 100$ the quantity

$$H_{\mathrm{rel}}^*(p) = 1 + \frac{1}{2}\min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi)l_0(\eta)\sum_{k=2}^{p}\kappa_k(\xi)\kappa_k(\eta)$$

was computed. For the sake of completness, we define $H_{\mathrm{rel}}^*(1) = 1$. The value of $H_{\mathrm{rel}}^*(p)$ can be found analytically for $2 \leq p \leq 4$. For $5 \leq p \leq 100$, it needs to be computed numerically. As it is seen from Table 1 and Figure 3, the worst case is $p = 3$ where $H_{\mathrm{rel}}^*(3) = 9/10$. Thus, the discrete Green's function is nonnegative.

6 Main Result

Let us summarize the conclusions of the previous analysis:

Theorem 6.1. *Let $\alpha = x_0 < x_1 < \ldots < x_M = \beta$ be a partition of the domain $\Omega = (\alpha,\beta)$ and let $p_i \geq 1$ be a polynomial degree assigned to the element $K_i = [x_{i-1}, x_i]$, $i = 1,2,\ldots,M$. If*

$$H_{\mathrm{rel}}^*(p_i) \geq 0 \quad \textit{for all } i = 1,2,\ldots,M, \tag{19}$$

then problem (2) satisfies the discrete maximum principle

p	$H^*_{\mathrm{rel}}(p)$	p	$H^*_{\mathrm{rel}}(p)$	p	$H^*_{\mathrm{rel}}(p)$	p	$H^*_{\mathrm{rel}}(p)$
1	1	6	1	11	0.953759	16	0.968695
2	1	7	0.935127	12	0.969485	17	0.967874
3	9/10	8	0.987060	13	0.959646	18	0.969629
4	1	9	0.945933	14	0.968378	19	0.970855
5	0.919731	10	0.973952	15	0.964221	20	0.970814

Table 1: The quantity $H^*_{\mathrm{rel}}(p)$ for $p = 1, 2, 3, \ldots, 20$.

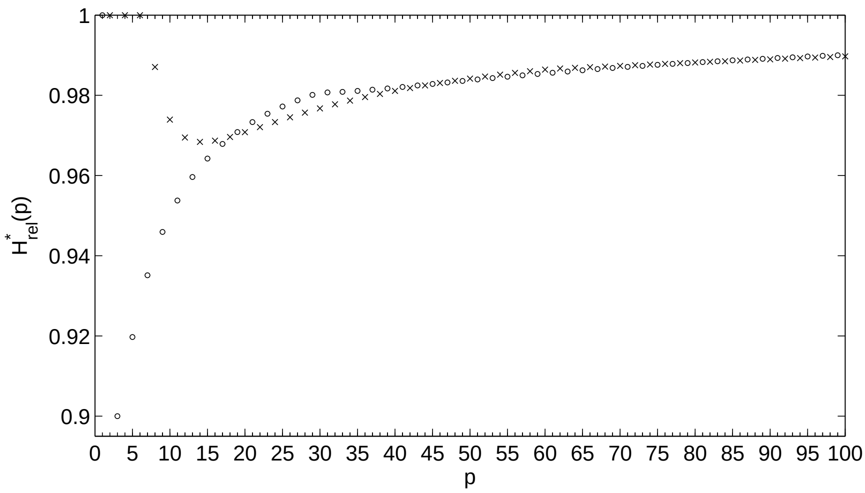

Figure 3: The values $H^*_{\mathrm{rel}}(p)$ for $p = 1, 2, \ldots, 100$. Circles indicate the values for p odd and crosses for p even.

Proof. Let K_i be an element. By (15), (18), and (19) it holds

$$G_{hp}(x, z)|_{K_i^2} = \hat{G}_{hp}(\xi, \eta) \geq 0$$

for all $[x, z] \in K_i^2$ with $\xi = \chi_{K_i}^{-1}(x)$ and $\eta = \chi_{K_i}^{-1}(z)$. Thus, $G_{hp}(x, z) \geq 0$ in $\bigcup_{i=1}^{M} K_i^2$. Lemma 4.2 implies that $G_{hp}(x, z) \geq 0$ also in $\Omega^2 \setminus \bigcup_{i=1}^{M} K_i^2$. Theorem 3.1 and Lemma 2.1 finish the proof. □

The crucial condition (19) was verified analytically for $p \leq 4$, therefore Theorem 6.1 proves the discrete maximum principle for problem (2) for all meshes and arbitrary polynomial degrees not exceeding 4. However, numerical calculations of $H^*_{\mathrm{rel}}(p)$ show that the condition (19) is satisfied for $5 \leq p \leq 100$ as well. Moreover, the steadily growing trend in H^*_{rel} for $p \geq 50$ observed in Figure 3 motivates the following conjecture:

Conjecture 6.1. *The problem* (2) *satisfies the discrete maximum principle for arbitrary partition of the domain* $\Omega = (\alpha, \beta)$ *and for arbitrary distribution of polynomial degrees.*

Appendix

The Lobatto shape functions are defined by

$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x, \quad j = 2, 3, \ldots,$$

where $P_j(x) = \mathrm{d}^j/\mathrm{d}x^j (x^2-1)^j/(2^j j!)$ stands for the jth-degree Legendre polynomial. The kernels are defined by $\kappa_j(\xi) = l_j(\xi)/(l_0(\xi)l_1(\xi))$, where $l_0(\xi) = (1-\xi)/2$, $l_1(\xi) = (1+\xi)/2$, and $\xi \in [-1, 1]$. These kernels can be generated by the recurrence

$$\kappa_{j+2}(\xi) = \frac{\sqrt{2j+1}\sqrt{2j+3}}{j+2}\xi\kappa_{j+1}(\xi) - \frac{j-1}{j+2}\sqrt{\frac{2j+3}{2j-1}}\kappa_j(\xi), \quad j = 2, 3, \ldots.$$

Interesting observation is that these kernels are scaled derivatives of Legendre polynomials

$$\kappa_j(\xi) = \sqrt{\frac{2j-1}{2}} \frac{4}{j(1-j)} P'_{j-1}(\xi), \quad j = 2, 3, \ldots.$$

Hence, they form a system of orthogonal polynomials with weight $1 - \xi^2 = 4l_0(\xi)l_1(\xi)$. For reference, we list several kernel functions κ_i (see, e.g., Section 3.1 in [17] or Section 1.2 in [15]):

$$
\begin{aligned}
\kappa_2(\xi) &= -\sqrt{6} \\
\kappa_3(\xi) &= -\sqrt{10}\xi \\
\kappa_4(\xi) &= -\frac{1}{4}\sqrt{14}(5\xi^2 - 1) \\
\kappa_5(\xi) &= -\frac{3}{4}\sqrt{2}(7\xi^2 - 3)\xi \\
\kappa_6(\xi) &= -\frac{1}{8}\sqrt{22}(21\xi^4 - 14\xi^2 + 1) \\
\kappa_7(\xi) &= -\frac{1}{8}\sqrt{26}(33\xi^4 - 30\xi^2 + 5)\xi \\
\kappa_8(\xi) &= -\frac{1}{64}\sqrt{30}(429\xi^6 - 495\xi^4 + 135\xi^2 - 5) \\
\kappa_9(\xi) &= -\frac{1}{64}\sqrt{34}(715\xi^6 - 1001\xi^4 + 385\xi^2 - 35)\xi \\
\kappa_{10}(\xi) &= -\frac{1}{128}\sqrt{38}(2431\xi^8 - 4004\xi^6 + 2002\xi^4 - 308\xi^2 + 7)
\end{aligned}
$$

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