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Discrete Maximum Principle for a 1D Problem with Piecewise-Constant Coefficients Solved by hp -FEM

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Abstract: *In this paper we prove the discrete maximum principle for a one-dimensional equation of the form $-(au')' = f$ with piecewise-constant coefficient $a(x)$, discretized by the hp-FEM. The discrete problem is transformed in such a way that the discontinuity of the coefficient $a(x)$ disappears. Existing results are then applied to obtain a condition on the mesh which guarantees the satisfaction of the discrete maximum principle. Both Dirichlet and mixed Dirichlet-Neumann boundary conditions are discussed.*

AMS subject classification: 65N30, 35B50

Keywords: Discrete maximum principle,, hp-FEM, Poisson equation, piecewise-constant coefficients.

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1 Introduction

Discrete maximum principles (DMP) have been studied since the 1970s (see, e.g., [3, 4] and the references therein). In particular, the maximum and comparison principles belong to the most important qualitative properties of numerical schemes. They guarantee, for example, the nonnegativity of approximations of naturally nonnegative quantities such as temperature, density, concentration, etc. When a numerical method does not satisfy the DMP, it can happen that the resulting numerical solution contradicts the physics. Therefore, the study of numerical methods equipped with discrete maximum principles became very popular during the last years.

Absolute majority of results about DMP concern lowest-order, such as piecewise-linear approximations [2, 5, 7, 8, 9, 10, 16, 17, 22]. Recent rapid development of higher-order methods, especially *hp*-FEM [1, 11, 12, 13, 15], leads to a question whether and under which conditions the DMP can be extended to this type of approximations. The answer to this question is difficult since there is no simple condition for polynomials to be nonnegative – in contrast to the lowest-order case. There only are a few works addressing DMP for higher-order approximations (see [6, 21] and recent results of the authors [14, 18, 19, 20]).

A particularly discouraging 2D result [6] shows that a *stronger* DMP for quadratic elements only is valid under prohibitively restrictive conditions on the mesh and that for higher-order elements the stronger DMP is not valid at all. The point is that the stronger DMP requires the maximum principle to be fulfilled on all subdomains, particularly on patches sharing common vertex. This requirement, however, is too strong and its relaxation leaves space for investigation of a condition for the mesh for the DMP on the whole domain.

Recently, discrete maximum principles for the Poisson equation in 1D, discretized by *hp*-FEM, were studied in [14, 18, 19, 20]. The present paper generalizes these results to the equation $-(au')' = f$, where the coefficient $a(x)$ is assumed to be discontinuous and piecewise-constant. We derive a sufficient condition on the mesh that guarantees the DMP in the case of Dirichlet boundary conditions. This condition involves the coefficient a and it can be easily verified in an element-by-element fashion. The case of mixed Dirichlet-Neumann boundary conditions is studied as well, and it is shown that the DMP is valid on all meshes with arbitrary distribution of polynomial degrees.

The scope of the paper is as follows: A model problem with piecewise-constant coefficient $a(x)$ and homogeneous Dirichlet boundary conditions is formulated in Section 2. In Section 3 we transform this problem to a new one with a constant coefficient $\tilde{a}(x) = 1$, so that its solution exactly coincides with

the solution to the original model problem. In Section 4 we infer a condition for DMP for the original model problem. Finally, Section 5 discusses the case of mixed Dirichlet-Neumann boundary conditions.

2 Model problem and its discretization

We solve the one-dimensional equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -(a(x)u(x)')' &= f(x) \quad \text{in } \Omega, \\ u(\alpha) &= u(\beta) = 0, \end{aligned}$$

in an interval $\Omega = (\alpha, \beta) \subset \mathbb{R}$. The corresponding weak formulation reads: Find $u \in V$ such that

$$\mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V, \quad (1)$$

where $V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v(\alpha) = v(\beta) = 0\}$, $a \in L^\infty(\Omega)$ is piecewise-constant, $f \in L^2(\Omega)$ is a right-hand side, and

$$(f, v)_\Omega = \int_\Omega f(x)v(x) dx, \quad \text{and} \quad \mathcal{B}(u, v) = \int_\Omega a(x)u'(x)v'(x) dx.$$

As usual, we create a partition $\alpha = x_0 < x_1 < \dots < x_M = \beta$ of the domain Ω consisting of M elements $K_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, M$. Every element K_i is assigned an arbitrary polynomial degree $p_i \geq 1$. Moreover, we assume the piecewise-constant coefficient a to be aligned with this partition. The corresponding finite element space of continuous and piecewise-polynomial functions $V_{hp} \subset V$ has the form

$$V_{hp} = \{v_{hp} \in V; v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \dots, M\}.$$

Here $P^{p_i}(K_i)$ stands for the space of polynomials of degree at most p_i on the element K_i . The space V_{hp} has the dimension $N = -1 + \sum_{i=1}^M p_i$. There exists unique finite element function $u_{hp} \in V_{hp}$ satisfying

$$\mathcal{B}(u_{hp}, v_{hp}) = (f, v_{hp}) \quad \forall v_{hp} \in V_{hp}. \quad (2)$$

3 Transformed problem

The coefficient $a = a(x)$ is considered to be piecewise-constant with respect to the partition of Ω , i.e., there exist constants a_i such that

$$a|_{K_i} = a_i \quad \forall i = 1, 2, \dots, M.$$

We will transform the model problem (1) to a standard Poisson equation in a different domain $\tilde{\Omega}$ and with a different right-hand side. The right-hand side \tilde{f} , the domain $\tilde{\Omega}$, and the partition of $\tilde{\Omega}$ will be determined later. The Poisson equation has the form

$$\begin{aligned} -\tilde{u}''(\tilde{x}) &= \tilde{f}(\tilde{x}) \quad \text{in } \tilde{\Omega}, \\ \tilde{u}(\tilde{\alpha}) &= \tilde{u}(\tilde{\beta}) = 0. \end{aligned}$$

The weak formulation reads: Find $\tilde{u} \in \tilde{V}$ such that

$$\tilde{\mathcal{B}}(\tilde{u}, \tilde{v}) = (\tilde{f}, \tilde{v})_{\tilde{\Omega}} \quad \forall \tilde{v} \in \tilde{V}, \quad (3)$$

where $\tilde{V} = \{\tilde{v} \in H^1(\tilde{\Omega}) : \tilde{v}(\tilde{\alpha}) = \tilde{v}(\tilde{\beta}) = 0\}$, $\tilde{f} \in L^2(\tilde{\Omega})$,

$$(\tilde{f}, \tilde{v})_{\tilde{\Omega}} = \int_{\tilde{\Omega}} \tilde{f}(\tilde{x})\tilde{v}(\tilde{x}) \, d\tilde{x}, \quad \text{and} \quad \tilde{\mathcal{B}}(\tilde{u}, \tilde{v}) = \int_{\tilde{\Omega}} \tilde{u}'(\tilde{x})\tilde{v}'(\tilde{x}) \, d\tilde{x}.$$

We construct a partition $\tilde{\alpha} = \tilde{x}_0 < \tilde{x}_1 < \dots < \tilde{x}_M = \tilde{\beta}$ of the domain $\tilde{\Omega}$ consisting of M elements $\tilde{K}_i = [\tilde{x}_{i-1}, \tilde{x}_i]$, $i = 1, 2, \dots, M$. Every element \tilde{K}_i is assigned a polynomial degree $p_i \geq 1$. Notice that the number of elements as well as the polynomial degrees are exactly the same as for the original problem (2).

The corresponding finite element space $\tilde{V}_{hp} \subset \tilde{V}$ is given by

$$\tilde{V}_{hp} = \left\{ \tilde{v}_{hp} \in \tilde{V}; \tilde{v}_{hp}|_{\tilde{K}_i} \in P^{p_i}(\tilde{K}_i), \quad i = 1, 2, \dots, M \right\}.$$

Clearly, $\dim \tilde{V}_{hp} = \dim V_{hp} = N$. Finally, the finite element solution $\tilde{u}_{hp} \in \tilde{V}_{hp}$ to problem (3) is uniquely given by requirement

$$\tilde{\mathcal{B}}(\tilde{u}_{hp}, \tilde{v}_{hp}) = (\tilde{f}, \tilde{v}_{hp})_{\tilde{\Omega}} \quad \forall \tilde{v}_{hp} \in \tilde{V}_{hp}. \quad (4)$$

Now let us link the discrete problem (2) to the discrete Poisson problem (4). There is a single degree of freedom which is the left endpoint $\tilde{\alpha} \in \mathbb{R}$, all remaining data to problem (4) are uniquely determined by the data to the original problem (2). First let us define the lengths of the new elements,

$$\tilde{h}_i = h_i/a_i, \quad i = 1, 2, \dots, M, \quad (5)$$

where $h_i = x_i - x_{i-1}$. The points of the new partition of $\tilde{\Omega}$ are given by

$$\tilde{x}_i = \tilde{\alpha} + \sum_{k=1}^i \tilde{h}_k, \quad i = 1, 2, \dots, M.$$

Moreover, we put $\tilde{x}_0 = \tilde{\alpha}$ and $\tilde{\beta} = \tilde{x}_M$. For future reference let us define affine transformations of elements K_i to elements \tilde{K}_i by

$$\eta_i(x) = \frac{\tilde{h}_i}{h_i}(x - x_{i-1}) + \tilde{x}_{i-1}, \quad i = 1, 2, \dots, M. \quad (6)$$

Finally, the right-hand side to the transformed problem (4) is defined in an element-by-element fashion as

$$\tilde{f}(\tilde{x})|_{\tilde{K}_i} = a_i f(x)|_{K_i},$$

where $x = \eta_i^{-1}(\tilde{x})$, $i = 1, 2, \dots, M$.

Our subsequent results are based on the following Lemma 3.1 and Theorem 3.1:

Lemma 3.1. *Let $u, v \in H^1(\Omega)$ and $\tilde{u}, \tilde{v} \in H^1(\tilde{\Omega})$ be functions satisfying*

$$u(x)|_{K_i} = \tilde{u}(\tilde{x})|_{\tilde{K}_i} \quad \text{and} \quad v(x)|_{K_i} = \tilde{v}(\tilde{x})|_{\tilde{K}_i},$$

where $\tilde{x} = \eta_i(x)$ and $i = 1, 2, \dots, M$. Then

$$\mathcal{B}(u, v) = \tilde{\mathcal{B}}(\tilde{u}, \tilde{v}) \quad \text{and} \quad (f, v)_\Omega = (\tilde{f}, \tilde{v})_{\tilde{\Omega}}.$$

Proof. Let us calculate

$$\mathcal{B}(u, v) = \sum_{i=1}^M a_i \int_{K_i} u'(x)v'(x) dx = \sum_{i=1}^M a_i \frac{\tilde{h}_i}{h_i} \int_{\tilde{K}_i} \tilde{u}'(\tilde{x})\tilde{v}'(\tilde{x}) d\tilde{x} = \tilde{\mathcal{B}}(\tilde{u}, \tilde{v}),$$

where we have used (6) for the substitution in the integral. Similarly,

$$(f, v)_\Omega = \sum_{i=1}^M \int_{K_i} f(x)v(x) dx = \sum_{i=1}^M \frac{h_i}{\tilde{h}_i} \frac{1}{a_i} \int_{\tilde{K}_i} \tilde{f}(\tilde{x})\tilde{v}(\tilde{x}) d\tilde{x} = (\tilde{f}, \tilde{v})_{\tilde{\Omega}}.$$

□

Theorem 3.1. *Let u_{hp} and \tilde{u}_{hp} be solutions to problems (2) and (4), respectively. Then*

$$u_{hp}(x)|_{K_i} = \tilde{u}_{hp}(\tilde{x})|_{\tilde{K}_i}, \quad (7)$$

where $\tilde{x} = \eta_i(x)$ and $i = 1, 2, \dots, M$.

Proof. Let $u_{hp} \in V_{hp}$ be the unique solution to (2). Let us use transformations (6) to define $\tilde{u}_{hp}^* \in \tilde{V}_{hp}$ as

$$\tilde{u}_{hp}^*(\tilde{x})|_{\tilde{K}_i} = u_{hp}(x)|_{K_i},$$

where $x = \eta_i^{-1}(\tilde{x})$ and $i = 1, 2, \dots, M$. Further, let $\tilde{v}_{hp} \in \tilde{V}_{hp}$ be arbitrary. Similarly, we define $v_{hp} \in V_{hp}$ by

$$v_{hp}(x)|_{K_i} = \tilde{v}_{hp}(\tilde{x})|_{\tilde{K}_i},$$

where $\tilde{x} = \eta_i(x)$ and $i = 1, 2, \dots, M$.

Now Lemma 3.1 and equality (2) imply

$$\tilde{\mathcal{B}}(\tilde{u}_{hp}^*, \tilde{v}_{hp}) = \mathcal{B}(u_{hp}, v_{hp}) = (f, v_{hp})_\Omega = (\tilde{f}, \tilde{v}_{hp})_{\tilde{\Omega}}.$$

Thus, $\tilde{u}_{hp}^* \in \tilde{V}_{hp}$ satisfies (4) for all $\tilde{v}_{hp} \in \tilde{V}_{hp}$. Since the solution $\tilde{u}_{hp} \in \tilde{V}_{hp}$ to problem (4) is unique, we have $\tilde{u}_{hp}^* = \tilde{u}_{hp}$ and the proof is finished. \square

4 Discrete maximum principle

In this section we will use existing results for the Poisson equation to infer a condition for the discrete maximum principle for the original problem (2). First let us recall the definition of the discrete maximum principle:

Definition 4.1. *Problem (2) satisfies the discrete maximum principle (DMP) if*

$$f \leq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad \max_{\tilde{\Omega}} u_{hp} = \max_{\partial\Omega} u_{hp},$$

where $\partial\Omega$ is the boundary of the domain Ω .

Let us define a fundamental quantity $H_{\text{rel}}^*(p)$:

$$H_{\text{rel}}^*(p) = 1 \quad \text{for } p = 1,$$

$$H_{\text{rel}}^*(p) = 1 + \frac{1}{2} \min_{(\xi, \eta) \in [-1, 1]^2} l_0(\xi)l_0(\eta) \sum_{k=2}^p \kappa_k(\xi)\kappa_k(\eta) \quad \text{for } p \geq 2.$$

Here, $l_0(\xi) = (1 - \xi)/2$ and $\kappa_k(\xi) = \sqrt{\frac{2k-1}{2}} \frac{4}{k(1-k)} P'_{k-1}(\xi)$, where $P_k(\xi)$ stand for the Legendre polynomials of degree k . See Table 1 for values $H_{\text{rel}}^*(p)$ for $1 \leq p \leq 20$. The values of $H_{\text{rel}}^*(p)$ up to $p = 100$ are depicted in Figure 1. Notice that the smallest value for $p \leq 100$ is $9/10$. This value can be calculated analytically.

Theorem 4.1. *Let $\tilde{\alpha} = \tilde{x}_0 < \tilde{x}_1 < \dots < \tilde{x}_M = \tilde{\beta}$ be a partition of the domain $\tilde{\Omega} = (\tilde{\alpha}, \tilde{\beta})$ and let $p_i \geq 1$ be polynomial degrees assigned to the elements $\tilde{K}_i = [\tilde{x}_{i-1}, \tilde{x}_i]$, $i = 1, 2, \dots, M$. If*

$$\frac{\tilde{x}_i - \tilde{x}_{i-1}}{\tilde{\beta} - \tilde{\alpha}} \leq H_{\text{rel}}^*(p_i) \quad \text{for all } i = 1, 2, \dots, M, \quad (8)$$

then problem (4) satisfies the discrete maximum principle.

p	$H_{\text{rel}}^*(p)$	p	$H_{\text{rel}}^*(p)$	p	$H_{\text{rel}}^*(p)$	p	$H_{\text{rel}}^*(p)$
1	1	6	1	11	0.953759	16	0.968695
2	1	7	0.935127	12	0.969485	17	0.967874
3	9/10	8	0.987060	13	0.959646	18	0.969629
4	1	9	0.945933	14	0.968378	19	0.970855
5	0.919731	10	0.973952	15	0.964221	20	0.970814

Table 1: The values of $H_{\text{rel}}^*(p)$ for $p = 1, 2, 3, \dots, 20$.

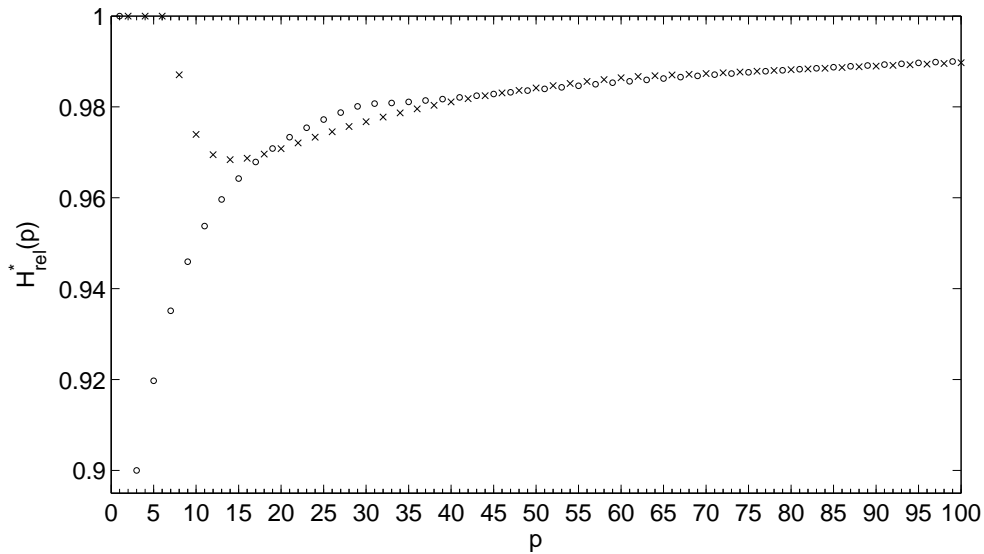


Figure 1: The values of $H_{\text{rel}}^*(p)$ for $p = 1, 2, \dots, 100$. Circles indicate the values for p odd and crosses for p even.

Proof. See [18]. □

We propose to generalize condition (8) to problems with piecewise-constant coefficient $a(x)$ in the following way:

Theorem 4.2. *Let $\alpha = x_0 < x_1 < \dots < x_M = \beta$ be a partition of the domain $\Omega = (\alpha, \beta)$, let $p_i \geq 1$ be polynomial degrees assigned to the elements $K_i = [x_{i-1}, x_i]$, let $h_i = x_i - x_{i-1}$, and let a_i stand for the constant values of a in K_i , $i = 1, 2, \dots, M$. If*

$$\frac{\overline{h_i}}{\sum_{k=1}^M \overline{h_k}} \leq H_{\text{rel}}^*(p_i) \quad \text{for all } i = 1, 2, \dots, M, \quad (9)$$

then problem (2) satisfies the discrete maximum principle.

Proof. This is a simple consequence of (5) and Theorems 3.1 and 4.1. \square

Note that condition (9) can easily be verified in an element-by-element fashion and that it can be written in a simple way using the notation from Section 3:

$$\frac{\tilde{h}_i}{\tilde{\beta} - \tilde{\alpha}} \leq H_{\text{rel}}^*(p_i) \quad \text{for all } i = 1, 2, \dots, M.$$

5 Mixed boundary conditions

Next let us consider the problem from Section 2, equipped with a Neumann boundary condition at β :

$$\begin{aligned} -(a(x)u(x)')' &= f(x) \quad \text{in } \Omega, \\ u(\alpha) &= 0, \\ u'(\beta) &= g(\beta). \end{aligned}$$

The weak formulation reads: Find $u \in V$ such that

$$\mathcal{B}(u, v) = (f, v)_\Omega + g(\beta)v(\beta) \quad \forall v \in V, \quad (10)$$

where $V = \{v \in H^1(\Omega) : v(\alpha) = 0\}$, $a \in L^\infty(\Omega)$ is piecewise-constant, and $f \in L^2(\Omega)$.

We proceed analogously to Section 2 to obtain the finite element solution $u_{hp} \in V_{hp}$,

$$\mathcal{B}(u_{hp}, v_{hp}) = (f, v_{hp})_\Omega + g(\beta)v(\beta) \quad \forall v_{hp} \in V_{hp}. \quad (11)$$

Now the dimension of the space $V_{hp} \subset V$ is greater by one compared to the space V_{hp} which was defined in Section 2. The original problem is transformed similarly to what was done in Section 3. The Neumann boundary data for the new problem are the same as for the original problem, i.e.,

$$\tilde{g}(\tilde{\beta}) = g(\beta).$$

Clearly, it follows from Lemma 3.1 that

$$(f, v)_\Omega + g(\beta)v(\beta) = (\tilde{f}, \tilde{v})_{\tilde{\Omega}} + \tilde{g}(\tilde{\beta})\tilde{v}(\tilde{\beta})$$

for $v(x)|_{K_i} = \tilde{v}(\tilde{x})|_{\tilde{K}_i}$, where $\tilde{x} = \eta_k(x)$ and $k = 1, 2, \dots, M$. Thus, analogously to Theorem 3.1 we obtain

$$u_{hp}(x)|_{K_i} = \tilde{u}_{hp}(\tilde{x})|_{\tilde{K}_i}.$$

The Neumann data $g(\beta)$ as well as the right-hand side f enter the definition of the discrete maximum principle for problem (11):

Definition 5.1. *Problem (11) satisfies the discrete maximum principle if*

$$f \leq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \leq 0 \quad \Rightarrow \quad \max_{\bar{\Omega}} u_{hp} = \max_{\partial\Omega} u_{hp}.$$

It was proven in [19] that the Poisson equation with mixed boundary conditions satisfies the DMP if

$$H_{\text{rel}}^*(p_i) \geq 0 \quad \text{for all } i = 1, 2, \dots, M. \quad (12)$$

This condition is problem-independent. Since the discrete solution to the transformed problem is equal to the discrete solution to the problem with piecewise-constant coefficient, we conclude that the DMP is valid for problem (11):

Theorem 5.1. *If condition (12) is satisfied, then problem (11) with mixed boundary conditions and piecewise-constant coefficient $a(x)$ satisfies the discrete maximum principle.*

Condition (12) is satisfied at least for all $p \leq 100$ (see Table 1 and Figure 1). Hence, the discrete maximum principle for 1D problems of type (10) with mixed boundary conditions and with piecewise-constant coefficient $a(x)$ are satisfied on arbitrary meshes and with arbitrary distribution of polynomial degrees (not exceeding 100).

Remark 5.1. *Above, homogeneous Dirichlet boundary conditions were considered for simplicity only. The result on the DMP holds for nonhomogeneous conditions as well. Indeed, we always can consider harmonic Dirichlet lift γ satisfying general Dirichlet boundary conditions and $-(a(x)\gamma'(x))' = 0$. To obtain the solution \hat{u}_{hp} to the problem with general Dirichlet boundary conditions, we just add this lift to the solution u_{hp} with homogeneous Dirichlet conditions, i.e., $\hat{u}_{hp} = \gamma + u_{hp}$. Notice that the lift γ satisfies the classical maximum principle. Thus, in the case of general Dirichlet boundary conditions, we have $u_{hp} = 0$ on $\partial\Omega$ and*

$$\max_{\bar{\Omega}}(u_{hp} + \gamma) \leq \max_{\bar{\Omega}} u_{hp} + \max_{\bar{\Omega}} \gamma = \max_{\partial\Omega} u_{hp} + \max_{\partial\Omega} \gamma = \max_{\partial\Omega}(u_{hp} + \gamma).$$

Hence, the solution \hat{u}_{hp} satisfies the DMP.

Moreover, in the piecewise-constant coefficient case, the lift γ is piecewise-linear and continuous, and it can be expressed explicitly as

$$\begin{aligned} \gamma(x) &= C_1 \int_{\alpha}^x 1/a(s) \, ds + C_2 \\ &= C_1 \left[\sum_{k=1}^{i-1} \frac{h_k}{a_k} + \frac{x - x_{i-1}}{a_i} \right] + C_2, \quad \text{for } x \in K_i, i = 1, 2, \dots, M, \end{aligned}$$

where C_1 and C_2 are integration constants to be determined from the boundary conditions. If a Dirichlet condition is prescribed at the left endpoint, i.e., $\hat{u}_{hp}(\alpha) = g_\alpha$, then $C_2 = g_\alpha$. Similarly, condition $\hat{u}_{hp}(\beta) = g_\beta$ implies $C_1 = (g_\beta - g_\alpha) / \left(\sum_{k=1}^M h_k / a_k \right)$. For the case of mixed boundary conditions, the Dirichlet lift is constant, i.e., $\gamma = g_\alpha$. This follows from the requirement $\gamma'(\beta) = 0$ which implies $C_1 = 0$. Since the lift γ is constant, we conclude that the DMP is valid even for the case of mixed and nonhomogeneous boundary conditions.

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