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Equidistributed Error Mesh for the Approximation of Exponential Boundary Layers

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Abstract: *In this paper we introduce a new finite element mesh for the approximation of exponential boundary layers in convection-diffusion problems of the form* $-\epsilon \Delta u - b \partial u / \partial x + cu = f$ *which are obtained by linearizing the Navier-Stokes equations. The new mesh is based on the equidistribution of interpolation error and it takes into account the finite computer arithmetic. It is demonstrated numerically that it leads to significantly better convergence rates compared to the widely-used Shishkin and Bakhvalov meshes.*

AMS subject classification: 35B50, 65N60

Keywords: Convection-diffusion equation, exponential boundary layer, optimal mesh.

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1 Introduction

Convection-diffusion problems of the form

$$\begin{aligned} -\epsilon\Delta u - b\frac{\partial u}{\partial x} + cu &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

which are obtained by linearizing the Navier-Stokes equations in a simplified setting, serve as a basis for the study of efficient numerical methods for the approximate solution of the Navier-Stokes equations. Among the most important goals is the design of optimal initial meshes for adaptive numerical schemes which suppress spurious oscillations in boundary layers. Equation (1) induces two types of boundary layers – parabolic and exponential. Efficient approximation of the parabolic boundary layers by *hp*-FEM was studied in [4]. For convergence studies of piecewise-linear/bilinear finite elements for problem (1) on Shishkin and Bakhvalov-type meshes see [1, 2, 3] and the references therein. In this paper we devise a close-to-optimal finite element mesh for the approximation of the exponential boundary layers by piecewise-linear elements.

We consider the model equation

$$-\epsilon u'' - bu' = 0 \quad \text{in } \Omega = (0,1) \tag{2}$$

where $u \in C^2(\Omega)$ and $f \in C(\Omega)$, equipped with boundary conditions,

$$u(0) = 0, \quad u(1) = 1. \tag{3}$$

Here, the positive constants b and ϵ represent the velocity and viscosity of the fluid, respectively. Equation (2) can be normalized by introducing $\tilde{\epsilon} = \epsilon/b$ and $\tilde{b} = 1$. Thus, without loss of generality we assume that $b = 1$ in the following, and $\epsilon > 0$ is a small constant such as $\epsilon = 10^{-3}, 10^{-4}, \ldots, 10^{-8}$.

The weak formulation of the problem is standard: Find $U \in V = H_0^1(\Omega)$ such that

$$a(U, v) = l(v) \quad \text{for all } \ v \in V. \tag{4}$$

The bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is defined as

$$a(U, v) = \int_\Omega (\epsilon U'v' - bU'v)\,\mathrm{d}x, \quad v \in V,$$

and the linear form $l \in V'$ reads

$$l(v) = \int_\Omega (\epsilon G'v' + bG'v)\,\mathrm{d}x, \quad v \in V.$$

Here, $G \in H^1(\Omega)$ is the standard Dirichlet lift representing the nonhomogeneous Dirichlet boundary condition (3) at $x = 1$. The existence and uniqueness of solution to the model problem follows easily from the Lax-Milgram lemma.

The exact solution to problem (2), (3) has the form

$$u(x) = \frac{e^{-\frac{x}{\epsilon}} - 1}{e^{-\frac{1}{\epsilon}} - 1}. \tag{5}$$

For illustration, the function $u(x)$ is shown in Fig 1 for a rather large viscosity value $\epsilon = 10^{-2}$:

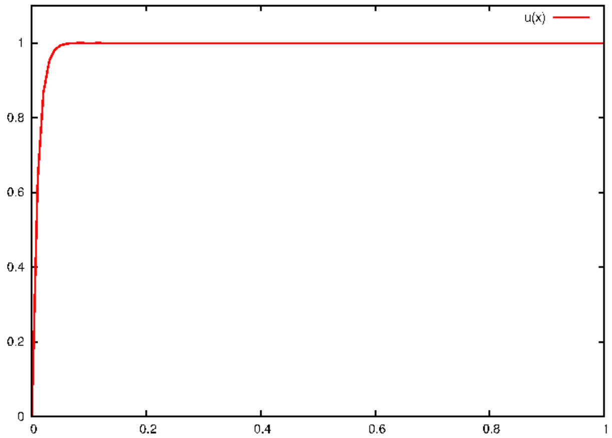

Figure 1: Exact solution $u(x)$ for $\epsilon = 10^{-2}$.

For smaller values of the viscosity, such as $\epsilon = 10^{-3}, 10^{-4}, \ldots$, the boundary layer at $x = 0$ shrinks quickly and it becomes invisible due to the finite graphical resolution.

2 Finite Computer Arithmetic and Adapted Equidistant Mesh

Let the interval Ω be covered with a finite element mesh $\mathcal{T}_h = \{K_1, K_2, \ldots, K_M\}$, $K_i = (x_{i-1}, x_i)$ where $0 = x_0 < x_1 < \ldots < x_M = 1$. If the division is equidistant, the mesh is called *equidistant (EQ) mesh.*

The mesh $\mathcal{T}_h$ induces the standard finite element space

$$V_h = \{v \in V;\ v|_{K_i} \in P^1(K_i)\} \subset V$$

of continuous, piecewise-linear functions in Ω. The finite element approximation of u in the space V_h is denoted by u_h.

In order to construct an optimal mesh for the approximation of problem (2), (3), we need to incorporate the finite computer arithmetic into our considerations. By $EPS > 0$ let us denote the machine zero of a computer, i.e., the largest real number such that

$$1 - |z| = 1 \quad \text{for all } |z| < EPS.$$

In particular, in this paper we use the value $EPS = 10^{-15}$ which corresponds to double precision computer accuracy.

Note that

$$e^{-35} < 10^{-15} < e^{-34},$$

and thus for $0 < \epsilon \leq 10^{-2}$, the exact solution (5) is represented in the computer as

$$u(x) = 1 - e^{-\frac{x}{\epsilon}}, \quad x \in \Omega. \tag{6}$$

In particular, it is

$$u(x) = 1 \quad \text{for all } x \in [35\epsilon, 1].$$

This fact can be used to save degrees of freedom in the interval $[35\epsilon, 1]$ by using grid points

$$0 = x_0 < x_1 < \ldots < x_{M-1} = 35\epsilon < x_M = 1. \tag{7}$$

If, moreover, the division is equidistant in the subinterval $(0, 35\epsilon)$, the mesh is called *adapted equidistant (AEQ) mesh.*

Fig. 2 presents a comparison of convergence rates of a piecewise-linear finite element method on EQ and AEQ meshes for the values $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$. The horizontal axis represents the number of degrees of freedom and the vertical shows the H^1-seminorm of the approximation error

$$e_h = u - u_h. \tag{8}$$

It follows from Fig. 2 that the incorporation of the finite computer arithmetic improves the convergence rates by roughly one order of magnitude.

Therefore, in what follows, the finite computer arithmetic will allways be taken into account.

3 Equidistributed Error (EE) Mesh

It is well-known that meshes with equidistributed approximation error yield excellent convergence rates. However, the task of constructing a finite element mesh $\mathcal{T}_h$ so that the approximation error (8) is distributed uniformly in all elements leads to a system of nonlinear algebraic equations whose exact

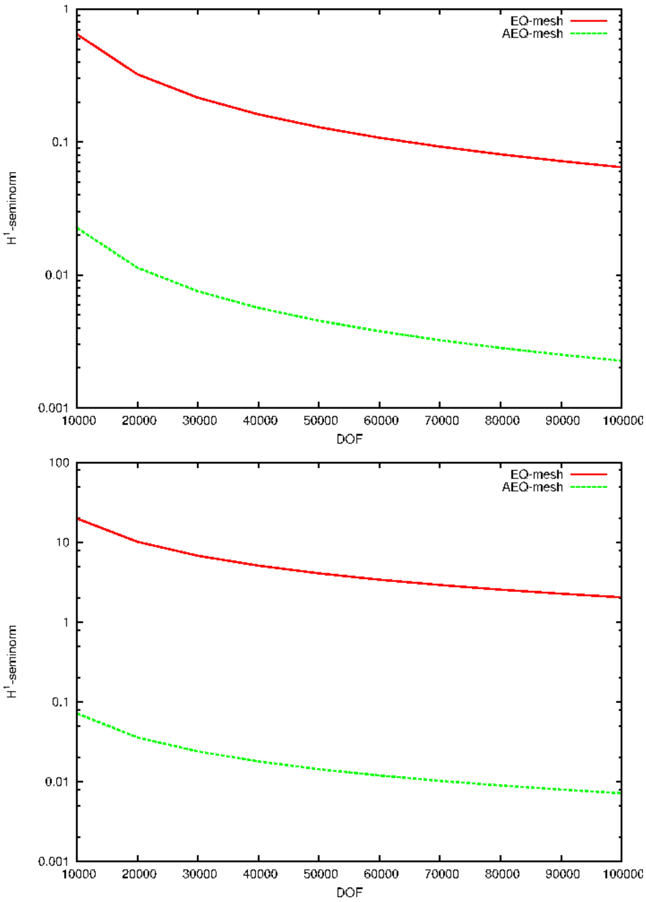

Figure 2: Convergence of piecewise-linear FEM on EQ and AEQ meshes: $\epsilon = 10^{-3}$ (left) and $\epsilon = 10^{-4}$ (right).

solution is extremely difficult to find. Therefore, we resort to an asymptotic argument which allows us to replace the approximation error with the *interpolation error*:

Given a mesh τ_h, let us define a piecewise-linear vertex interpolant $u^v \in V_h$ of the exact solution u such that

$$u^v(x_i) = u(x_i), \quad \text{for all } i = 0, 1, \ldots, M.$$

3.1 Calculation of the Interpolation Error

Given a grid point $x_i \in [0, 35\epsilon]$, we want to find a formula for the error of the piecewise-linear interpolant,

$$|u - u^v|_{H^1(x_i, x_{i+1})},$$

as a function of the position of the next grid point x_{i+1}. For this purpose, we denote $h = x_{i+1} - x_i$. From (6) we have that

$$u(x_i) = 1 - e^{-\frac{x_i}{\epsilon}}, \qquad u(x_i + h) = 1 - e^{-\frac{x_i+h}{\epsilon}}.$$

Then the slope m_i of the interpolant $u^v(x)$ in the interval (x_i, x_{i+1}) is

$$m_i = \frac{e^{-\frac{x_i+h}{\epsilon}} - e^{-\frac{x_i}{\epsilon}}}{h} = e^{-\frac{x_i}{\epsilon}} \left(\frac{1 - e^{-\frac{h}{\epsilon}}}{h} \right).$$

After some calculation we obtain

$$|u-u^v|^2_{H^1(x_i,x_{i+1})} = e^{-\frac{2x_i}{\epsilon}} \left[\frac{1}{2\epsilon} \left(1 - e^{-\frac{2h}{\epsilon}}\right) - \frac{1}{h} \left(1 - e^{-\frac{h}{\epsilon}}\right)^2 \right] = e^{-\frac{2x_i}{\epsilon}} f(h), \quad (9)$$

i.e.,

$$|u - u^v|^2_{H^1(x_i,x_{i+1})} = e^{-\frac{2x_i}{\epsilon}} f(h). \quad (10)$$

The function $f(h)$ is increasing monotonically from zero to its limit $1/(2\epsilon)$ for $h \to \infty$, as shown in Figure 3.

3.2 Construction of the EE-Mesh

To generate the equidistributed error (EE) mesh, we begin with choosing the first element $K_1 = (x_0, x_1)$. This determines the interpolation error E_0 for all elements in the mesh,

$$E_0 = f(h_0).$$

Then we proceed according to the following algorithm:

1. Having constructed the element K_i, the length h_{i+1} of the next element is obtained by solving the nonlinear equation
$$f(h_{i+1}) = E_0 e^{\frac{2x_i}{\epsilon}}. \quad (11)$$
2. The next grid point x_{i+1} is defined as
$$x_{i+1} = x_i + h_{i+1}.$$
3. If
$$x_{i+1} \leq -\frac{\epsilon}{2} \ln(2\epsilon E_0), \quad (12)$$
then increase i and go to step 1. Otherwise stop.

The stopping criterion (12) is based on the fact that if $x_i > -\frac{\epsilon}{2} \ln(2\epsilon E_0)$, then the error until the end of Ω is less than E_0.

The last grid point that we find with the algorithm above is denoted by x_{M-1}. The EE-mesh is completed by defining $K_{M-1} = (x_{M-1}, 1)$.

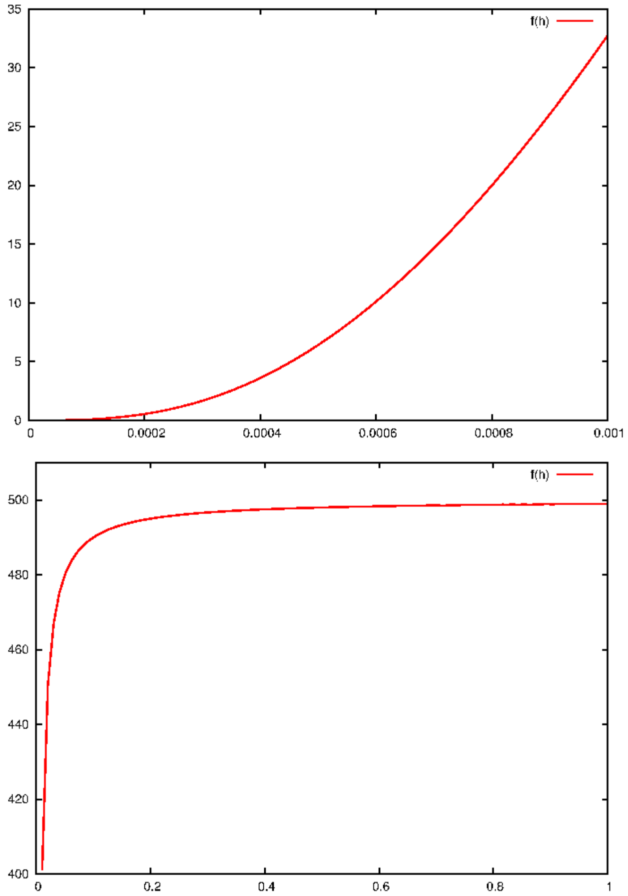

Figure 3: Graph of $f(h)$ for $\epsilon = 10^{-3}$.

4 Comparison to Shishkin and Bakhvalov Meshes

4.1 Layer-Adapted Shishkin Mesh

In agreement to the previous considerations on finite computer accuracy, we adjust the standard definition of the Shishkin mesh [3] as follows:

$$x_i = \begin{cases} \sigma\epsilon\varphi(t_i) \quad \text{with} \quad t_i = i/M & \text{for } i = 0, 1, \ldots, \frac{1}{2}M, \\ 35\epsilon - (35\epsilon - x_{N/2}) \cdot 2(M-i)/M & \text{for } i = \frac{1}{2}M + 1, \ldots, M. \end{cases}$$

Here, $\varphi(t)$ is a mesh generation function,

$$\varphi(t) = 2(\ln M)t \qquad \text{with} \quad \lambda = \sigma\epsilon \ln M,$$

and $\sigma > 0$ a real parameter. The choice of the parameter σ depends on the problem solved. In order to select the best value of σ for our problem, we produced a series of convergence curves for various values of σ:

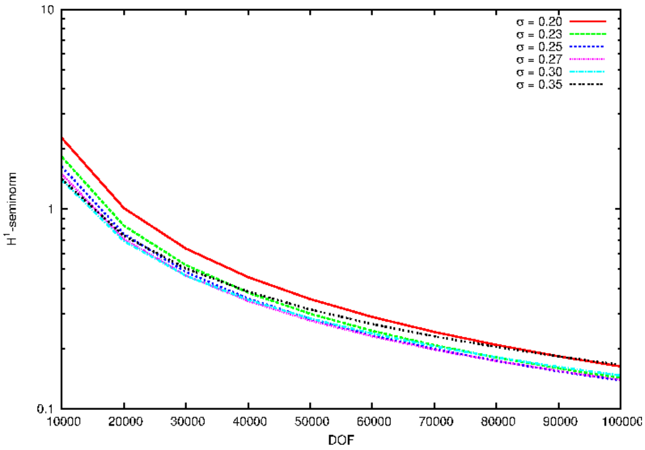

Figure 4: H^1-seminorm of the approximation error $e_h = u - u_h$ for various values of σ with $\epsilon = 10^{-8}$.

The curves shown in Fig. 4 correspond to $\epsilon = 10^{-8}$, but they were very similar also for all other considered values of ϵ, and we could safely conclude that the value $\sigma = 0.25$ yields fastest convergence of the finite element method on the Shishkin meshes for our problem.

4.2 Layer-Adapted Bakhvalov Mesh

Taking into account the finite computer arithmetic, we adjust the standard definition of the Bakhvalov mesh [3] to

$$x_i = \begin{cases} \sigma\epsilon\varphi(t_i) \text{ with } t_i = i/M & \text{for } i = 0, 1, \ldots, \frac{1}{2}M, \\ 35\epsilon - (35\epsilon - x_{M/2}) \cdot 2(M-i)/M & \text{for } i = \frac{1}{2}M + 1, \ldots, M, \end{cases}$$

where again $\varphi(t)$ is a mesh generating function,

$$\varphi(t) = -\ln[1 - 2(1-\epsilon)t].$$

Analogously to the previous case, we find that the optimal value of the parameter σ for our problem is $\sigma = 1.5$ (see Fig. 5).

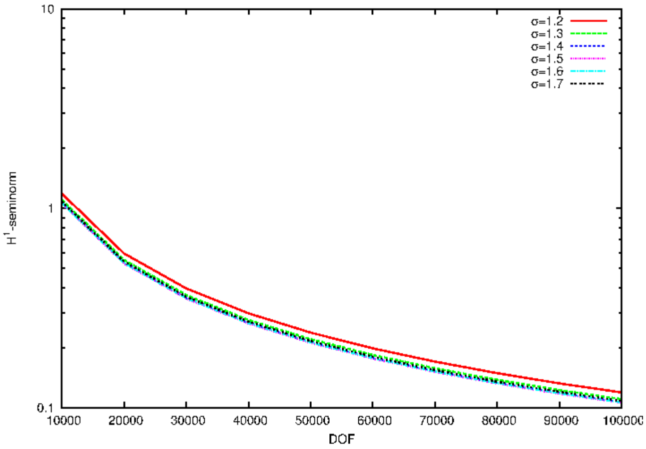

Figure 5: H^1-seminorm of the approximation error $e_h = u - u_h$ for various values of σ with $\epsilon = 10^{-8}$.

4.3 Comparison

Performance of the adapted equidistant, Shishkin, Bakhvalov, and equidistributed error meshes for various values of ϵ is compared in Figs. 6 – 9.

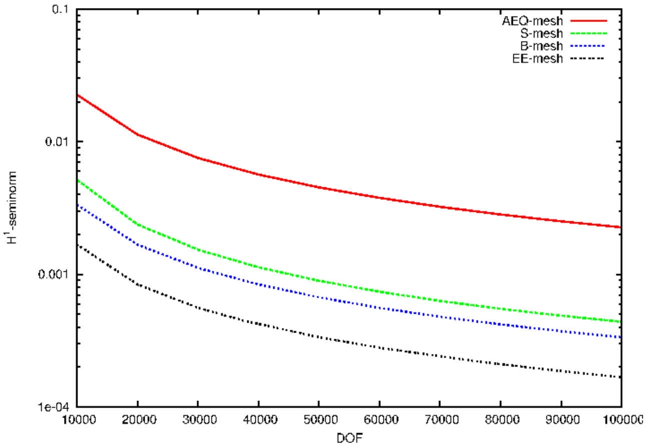

Figure 6: Convergence comparison, $\epsilon = 10^{-3}$.

5 Conclusion and Outlook

We derived a new equidistributed-interpolation error mesh (EE-mesh) for the approximation of exponential boundary layers in convections-diffusion problems. Numerical examples indicate that this mesh has significantly better approximation properties than the widely-used Shishkin and Bakhvalov

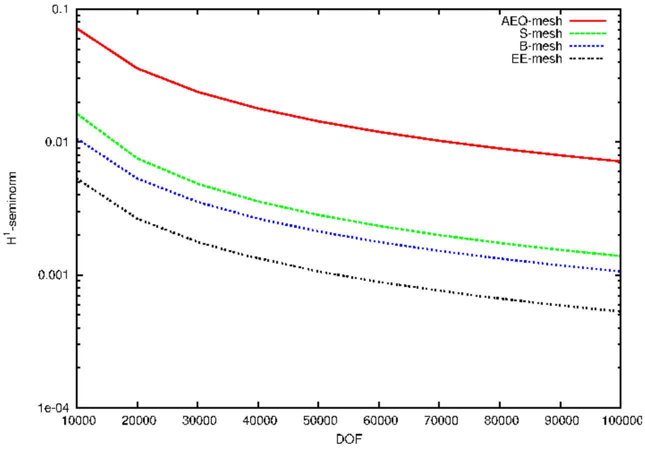

Figure 7: Convergence comparison, $\epsilon = 10^{-4}$.

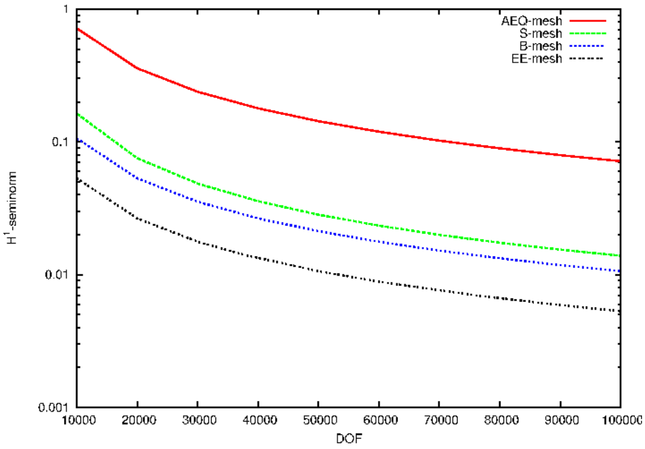

Figure 8: Convergence comparison, $\epsilon = 10^{-6}$.

meshes. Convergence analysis on the EE-mesh currently is in progress, as well as the application of the mesh to approximate boundary layers in two-dimensional Navier-Stokes equations.

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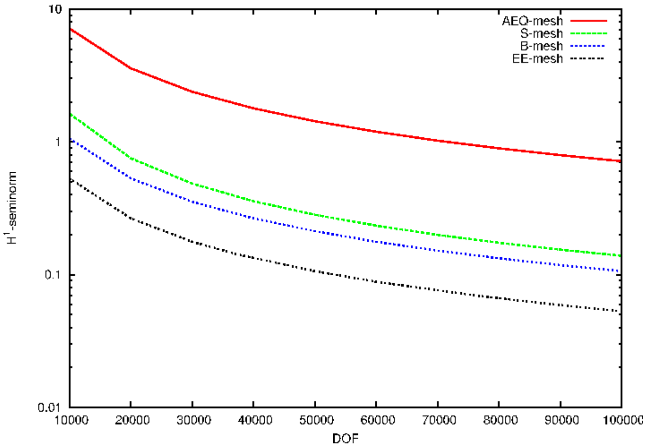

Figure 9: Convergence comparison, $\epsilon = 10^{-8}$.

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