6) Use the disk method to verify that the volume of a right circular cone is \( \frac{1}{3} \pi r^2 h \), where \( r \) is the radius of the base and \( h \) is the height.

Solution: Refer to the drawing at right. The cone has height \( h \) and radius \( r \). We draw a representative disk that has radius \( R \) and width \( \Delta y \). The height of our representative disk is \( y \). The volume of the representative disk is \( V_{\text{disk}} = \pi R^2 \Delta y \). We need to have \( R \) in terms of \( y \), so we must find the relationship between \( R \) and \( y \), that is, find \( R(y) \). As the drawing suggests, \( R \) is a linear function of \( y \), so \( R(y) = my + b \). We know that \( R(0) = r \) and \( R(h) = 0 \). Thus, \( m = \frac{\Delta R}{\Delta y} = \frac{r - 0}{0 - h} = -\frac{r}{h} \). The function is \( R(y) = -\left(\frac{r}{h}\right)y + r \).

The total volume is

\[
V_{\text{total}} = \pi \int_0^h [R(y)]^2 dy = \pi \int_0^h \left[-\left(\frac{r}{h}\right)y + r\right]^2 dy
\]

\[
= \pi \int_0^h \left(\frac{r}{h}\right)^2 y^2 - \left(\frac{2r^2}{h}\right)y + r^2 \right) dy = \pi \left[\frac{1}{3} \left(\frac{r^2}{h^2}\right)y^3 - \left(\frac{r^2}{h}\right)y^2 + r^2y\right]_0^h
\]

\[
= \pi \left[\frac{r^2}{3h^2} h^3 - \left(\frac{r^2}{h}\right)h^2 + r^2h \right] = \pi \left[\frac{1}{3} r^2h - r^2h + r^2h \right] = \frac{1}{3} \pi r^2 h
\]