1) Give a set-theoretic description of the given points as a subset $W$ of $\mathbb{R}^3$.

a) The points on the plane $x + y - 2z = 0$.

Solution: $W = \{x: x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 - 2x_3 = 0\}$.

b) The points in the $yz$-plane.

Solution: $W = \{x: x = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}, x_2, x_3 \in \mathbb{R}\}$.

2) In each case, determine whether $W$ is a subspace of $\mathbb{R}^3$.

a) $W = \{x: x_1 = 2x_2\}$

Solution: Check three things:

i. $\vec{0} \in W$ since $0 = 2(0)$.

ii. Let $\vec{u}, \vec{v} \in W$, where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$ where $u_1 = 2u_2$ and $v_1 = 2v_2$. Thus, $u_1 + v_1 = 2u_2 + 2v_2 = 2(u_2 + v_2)$. So, the first entry in $\vec{u} + \vec{v}$ is twice the second, and therefore $\vec{u} + \vec{v} \in W$.

iii. Finally, $a\vec{u} = a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}$, where $au_1 = a(2u_2) = 2(au_2)$. The first entry in $a\vec{u}$ is twice the second, so therefore $a\vec{u} \in W$.

Since $\vec{0} \in W$, $\vec{u} + \vec{v} \in W$, and $a\vec{u} \in W$, we can say that $W$ is a subspace of $\mathbb{R}^2$.

b) $W = \{x: x_1^2 + x_2 = 1\}$.

$W$ is not a subspace since $\vec{0} \not\in W$ ($0^2 + 0 \neq 1$).

3) In each case, determine whether $W$ is a subspace of $\mathbb{R}^3$.

a) $W = \{x: x_3 = 2x_1 - x_2\}$

Solution: Check three things:

i. $\vec{0} \in W$ since $0 = 2(0) - 0$.

ii. Let $\vec{u}, \vec{v} \in W$, where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$ where $u_3 = 2u_1 - u_2$ and $v_3 = 2v_1 - v_2$. Thus, $u_3 + v_3 = 2u_1 - u_2 + 2v_1 - v_2 = 2(u_1 + v_1) - (u_2 + v_2)$. So, the third entry in $\vec{u} + \vec{v}$ is twice the first minus the second. Therefore, $\vec{u} + \vec{v} \in W$.

iii. Finally, $a\vec{u} = a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}$, where $au_3 = a(2u_1 - u_2) = 2(au_1) - au_2$. The third entry in $a\vec{u}$ is twice the first minus the second, so therefore $a\vec{u} \in W$.

Since $\vec{0} \in W$, $\vec{u} + \vec{v} \in W$, and $a\vec{u} \in W$, we can say that $W$ is a subspace of $\mathbb{R}^3$. 

b) \( W = \{ \mathbf{x} : x_1 x_2 = x_3 \} \).

Solution: \( W \) is not a subspace. Consider \( \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \). Clearly, both \( \mathbf{u} \) and \( \mathbf{v} \) are elements of \( W \), however \( \mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \notin W \) since \( (3)(3) \neq 5 \).

c) \( W = \{ \mathbf{x} : x_1 = 2x_3, x_2 = -x_3 \} \)

Solution: Check three things:

i. \( \theta \in W \) since \( 0 = 2(0) \) and \( 0 = -0 \).

ii. Let \( \mathbf{u}, \mathbf{v} \in W \), where \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). Then \( \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \) where \( u_1 = 2u_3 \), \( u_2 = -u_3 \), and \( v_1 = 2v_3, v_2 = -v_3 \). Thus, \( u_1 + v_1 = 2u_3 + 2v_3 = 2(u_3 + v_3) \), and \( u_2 + v_2 = -u_3 - v_3 = -(u_3 + v_3) \). So, the first entry in \( \mathbf{u} + \mathbf{v} \) is twice the third, and the second entry is the negative of the third. Therefore, \( \mathbf{u} + \mathbf{v} \in W \).

iii. Finally, \( a\mathbf{u} = a \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix} \), where \( au_1 = a(2u_3) = 2(au_3) \) and \( au_2 = a(-u_3) = -au_3 \). The first entry in \( a\mathbf{u} \) is twice the third, and the second entry is the negative of the third. Therefore \( a\mathbf{u} \in W \).

Since \( \theta \in W \), \( \mathbf{u} + \mathbf{v} \in W \), and \( a\mathbf{u} \in W \), we can say that \( W \) is a subspace of \( \mathbb{R}^3 \).

4) Let \( U \) and \( V \) be subspaces of \( \mathbb{R}^n \). Prove that the intersection, \( U \cap V \), is also a subspace of \( \mathbb{R}^n \).

Solution: \( U \cap V = \{ \mathbf{x} : \mathbf{x} \in U \text{ and } \mathbf{x} \in V \} \). Let \( \mathbf{x}, \mathbf{y} \in U \cap V \). Then \( \mathbf{x}, \mathbf{y} \in U \) and \( \mathbf{x}, \mathbf{y} \in V \). Thus, \( \mathbf{x} + \mathbf{y} \in U \cap V \). Further, \( a\mathbf{x} \in U \) and \( a\mathbf{x} \in V \) (again since \( U \) and \( V \) are subspaces), so \( a\mathbf{x} \in U \cap V \). Thus, \( U \cap V \) is a subspace.

5) Given \( \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \), \( \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \), \( \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \). Let \( S = \{ \mathbf{w}, \mathbf{x}, \mathbf{y} \} \). Give an algebraic specification for \( \text{Sp}(S) \).

Solution: The span of \( S \) is the set of all linear combinations of the vectors in \( S \). To find an algebraic specification for \( \text{Sp}(S) \), we must solve \( c_1 \mathbf{w} + c_2 \mathbf{x} + c_3 \mathbf{y} = \mathbf{b} \). This is the same as solving \( A\mathbf{c} = \mathbf{b} \), where \( A = [\mathbf{w}, \mathbf{x}, \mathbf{y}] \) and \( \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \). This gives \( \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \), which reduces to \( \begin{bmatrix} 1 & -1 & 2 & b_3 \\ 0 & 1 & 1 & b_1 \\ 0 & 0 & 1 & \frac{1}{2}(b_2 + b_3) \end{bmatrix} \). This matrix is consistent for all \( \mathbf{b} \in \mathbb{R}^3 \), thus \( \text{Sp}(S) = \mathbb{R}^3 \).
6) Give an algebraic specification for the null space and the range of the given matrix $A$.

Solution (1st matrix): $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \end{bmatrix}$. For the null space, solve $Ax = \theta$. This gives $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which means $x_1 = -2x_2$ and $x_3 = 0$. Thus, $\mathcal{N}(A) = \{x: x = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}, x_2 \in \mathbb{R}\}$. For the range, solve $Ax = b$. That is, reduce the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 4 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 2 & 4b_1 - b_2 \\ 0 & 0 & 1 - 3b_1 + b_2 \end{bmatrix}$. Since this matrix is consistent for all $b \in \mathbb{R}^2$, we can conclude that $\mathcal{R}(A) = \mathbb{R}^2$.

For the range, you can also notice that $A$ is a $2 \times 3$ matrix, so $n = 3$. Also, notice that $\text{Null}(A) = 1$ since there is only 1 basis vector (for example, $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$) in $\mathcal{N}(A)$. Further, since $\text{Null}(A) + \text{Rank}(A) = 3$, this means that $\text{Rank}(A) = 2$. Any vector $b \in \mathcal{R}(A)$ must be an element of $\mathbb{R}^2$, and since $\text{Rank}(A) = 2$, we can conclude that $\mathcal{R}(A) = \mathbb{R}^2$.

2nd matrix: $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 5 \end{bmatrix}$. For the null space, solve $Ax = \theta$. This gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which means $x_1 = x_2 = 0$. Only $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{N}(A)$, so $\mathcal{N}(A) = \{\theta\}$. For the range, solve $Ax = b$. That is, reduce the matrix $\begin{bmatrix} 1 & 3 & b_1 \\ 2 & 7 & b_2 \\ 1 & 5 & b_3 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 3b_1 - 2b_2 + b_3 \end{bmatrix}$. This matrix is only consistent if $3b_1 - 2b_2 + b_3 = 0$. Thus, $\mathcal{R}(A) = \{x: x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, 3x_1 - 2x_2 + x_3 = 0\}$.

3rd matrix: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 3 & 4 \end{bmatrix}$. For the null space, solve $Ax = \theta$. This gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which means $x_1 = x_2 = x_3 = 0$. Only $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{N}(A)$, so $\mathcal{N}(A) = \{\theta\}$. Here, we have a $3 \times 3$ matrix, so $n = 3$. Since $\text{Null}(A) = 0$, this means $\text{Rank}(A) = 3$. Any vector $b \in \mathcal{R}(A)$ must be an element of $\mathbb{R}^3$, so we can conclude that $\mathcal{R}(A) = \mathbb{R}^3$.

7) This question is the same as Problem 2 from section 3.4. You can find the detailed solution here.
8) For the given set $S$:
   a) Find a subset $W$ of $S$ that is a basis for $\text{Sp}(S)$ using the first technique to find a basis.
   b) Find a basis for $\text{Sp}(S)$ using the second technique to find a basis.

Solution: For part a), the first technique to find a basis requires that we reduce the matrix $[A|\theta]$, where $A$ is the matrix with the four given vectors as its columns. This gives

$$
\begin{bmatrix}
1 & -2 & -1 & 0 \\
2 & 0 & -1 & 2 \\
-1 & 1 & 2 & 0 \\
0 & 2 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Since the matrix reduces to $[I|\theta]$, there are no unconstrained variables and therefore the given set $S$ is linearly independent and is therefore already a basis for $\text{Sp}(S)$.

For part b), reducing $A^T$ will give the identity matrix $I$ (which is its own transpose), so another basis for $\text{Sp}(S)$ is $W = \{e_1, e_2, e_3, e_4\}$.

9) Given $w = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Let $S = \{w, x, y\}$, determine whether $S$ is a basis for $\mathbb{R}^3$.

Solution: For a matrix $A = [w, x, y]$ and check to see if it is row equivalent to $I$. If it is, then the vectors in $S$ are linearly independent, and thus for a basis for $\mathbb{R}^3$ (because there are three of them). Reducing the matrix $A$ does indeed give $I$, so $S$ is a basis for $\mathbb{R}^3$.

10) Find a basis for $\mathcal{N}(A)$, $\mathcal{R}(A)$, and give the nullity and rank of $A$.

Solution: For each matrix, we can find a basis for the null space by reducing the matrix $[A|\theta]$.

For the first matrix, this gives

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \text{ so } x_1 = -x_3 - x_4 \text{ and } x_2 = -x_3 + x_4. \text{ The null space is } \mathcal{N}(A) = \left\{ x : x = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathbb{R} \right\}. \text{ If we let } u_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{ then a basis for } \mathcal{N}(A) \text{ is } \{u_1, u_2\} \text{. For a basis for the range, remember that the range is the same as the span of the vectors that make up the columns of } A, \text{ so we can find a basis for the range in the same way that we find a basis for the span of a set of vectors. That is, we can either reduce the matrix } [A|\theta] \text{ and delete the columns that correspond to unconstrained variables, or we can reduce } A^T \text{ to get a matrix } B^T \text{ and then use the nonzero columns of } B \text{ for our basis. Since we have already reduced the matrix } [A|\theta], \text{ we can see that the variables } x_3 \text{ and } x_4 \text{ are unconstrained, so we can delete columns 3 and 4 from the matrix } A \text{ and }$
get a basis for the range of \[
\begin{bmatrix}
1 & 3 \\
1 & 5 \\
1 & 1
\end{bmatrix}
\]. Since there are two vectors in the basis for the null space and 2 in the basis for the range, this means \(\text{Rank}(A) = \text{Nullity}(A) = 2\).

For the second matrix, reducing \([A|\theta]\) gives \([I|\theta]\). This means that the equation \(Ax = \theta\) has only the trivial solution, and therefore \(N(A) = \{\theta\}\), so \(\text{Nullity}(A) = 0\). This means \(\text{Rank}(A) = 4\) since \(A\) is a \(4 \times 4\) matrix. Since there are four columns in \(A\), this means that the four vectors that make up the columns of \(A\) must span \(\mathbb{R}^4\), and therefore form a basis for \(\mathcal{R}(A)\).

11) Let \(W\) be a subspace, and let \(S\) be a spanning set for \(W\). Find a basis for \(W\), and calculate \(\dim W\).
   Solution: To find a basis, reduce the matrix \([A|\theta]\), where \(A\) is the matrix whose columns are formed by the vectors in \(S\). This gives \([I|\theta]\), and so the vectors in \(S\) are linearly independent and thus form a basis for \(W\).

12) Use the Gram-Schmidt process to generate and orthogonal set from the given linearly independent vectors.
   Solution: For the first set of vectors,
   \[
   \begin{align*}
   u_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\
   u_2 &= w_2 - \frac{u_1^T w_2}{u_1^T u_1} u_1 = \begin{bmatrix} 0 \\ 2/2 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
   \\
   u_3 &= w_3 - \frac{u_1^T w_3}{u_1^T u_1} u_1 - \frac{u_2^T w_3}{u_2^T u_2} u_2 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{3} \frac{1}{1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}
   \end{align*}
   \]
   For the second set of vectors,
   \[
   \begin{align*}
   u_1 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \\
   u_2 &= w_2 - \frac{u_1^T w_2}{u_1^T u_1} u_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{3}{2} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}, \text{ and eliminating fractions gives } u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},
   \\
   u_3 &= w_3 - \frac{u_1^T w_3}{u_1^T u_1} u_1 - \frac{u_2^T w_3}{u_2^T u_2} u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{4} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{6} \frac{2}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \text{ or without fractions, } u_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}
   \end{align*}
   \]

13) This is the same as problem 5 on section 3.6. The solution is here.

We covered the problems from section 3.7 in class today (7/22), so see your notes for the solutions to problems 1, 3, and 4 from that section.