Problem 2 solution, 1st matrix:

a) Given matrix reduces to \( B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \).

b) To find a basis for the null space, solve \( Ax = \theta \). We can use \( B \) to get \( x_1 = -x_3, x_2 = -x_3, \) so
\[
N(A) = \left\{ x: x = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, x_3 \in \mathbb{R} \right\}.
\]
Thus, a basis for the null space is \( \{u\} \), where
\[
u = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.
\]

c) Recall that \( R(A) \) is the same thing as \( \text{Sp}\{A_1, A_2, A_3\} \), where \( A_1, A_2, A_3 \) are the columns of matrix \( A \). So, we can follow the technique given on the worksheet (and in the textbook) for finding a basis for the span of a set of vectors. Following this technique, a basis for \( R(A) \) consists of the first two columns of \( A \), since \( x_3 \) was an unconstrained variable in the solution of \( Ax = \theta \). Thus, a basis for \( R(A) \) is \( \{A_1, A_2\} \).

If we let \( x_3 = 1 \), then \( x_1 = x_2 = -1 \), then a solution to
\[
x_1A_1 + x_2A_2 + x_3A_3 = \theta \text{ is } -A_1 - A_2 + A_3 = \theta, \text{ so } A_3 = A_1 + A_2.
\]

d) According to the theorem given on the worksheet (and in the textbook), if a nonzero matrix \( A \) is row equivalent to the matrix \( B \) in echelon form, then the nonzero rows of \( B \) form a basis for the row space of \( A \). Thus, a basis for the row space of \( A \) is \( \{u_1, u_2\} \), where \( u_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \).

e) To use the technique of the row space to find a basis for \( R(A) \), we perform row operations on \( A^T \) until we reach reduced echelon form. Doing so gives the matrix \( B^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \), which
\[
B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ means } B^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{, which}
\]
means \( B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). Thus, a basis for \( R(A) \) is \( \{u_1, u_2\} \), where \( u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).
Problem 2 solution, 2nd matrix:

a) Given matrix reduces to \( B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

b) To find a basis for the null space, solve \( Ax = \mathbf{0} \). We can use \( B \) to get \( x_1 = x_3 - 2x_4 \),
\[
\begin{align*}
x_2 &= -x_3 + x_4, \text{ so } \mathcal{N}(A) = \left\{ x : x = \begin{bmatrix} x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathbb{R} \right\}. \text{ Thus, a basis for the null space is } \{u_1, u_2\} \text{ where } u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}. 
\end{align*}
\]

c) \( \mathcal{R}(A) \) is the same thing as \( \text{Sp}\{A_1, A_2, A_3, A_4\} \), where \( A_1, A_2, A_3, A_4 \) are the columns of matrix \( A \). Following the technique given in the textbook, a basis for \( \mathcal{R}(A) \) consists of the first two columns of \( A \), since \( x_3 \) and \( x_4 \) were unconstrained variables in the solution of \( Ax = \mathbf{0} \). Thus, a basis for \( \mathcal{R}(A) \) is \( \{A_1, A_2\} \).

If we let \( x_3 = 0, x_4 = 1 \), then \( x_1 = -2 \) and \( x_2 = 1 \), then a solution to
\[
x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = \mathbf{0}
\]
is \( -2A_1 + A_2 + A_4 = \mathbf{0} \), so \( A_4 = 2A_1 - A_2 \).
Now, if we let \( x_3 = 1, x_4 = 0 \), then \( x_1 = 1 \) and \( x_2 = -1 \), then a solution to
\[
x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = \mathbf{0}
\]
is \( A_1 - A_2 + A_3 = \mathbf{0} \), so \( A_3 = -A_1 + A_2 \).

d) According to the theorem given on the worksheet (and in the textbook), if a nonzero matrix \( A \) is row equivalent to the matrix \( B \) in echelon form, then the nonzero rows of \( B \) form a basis for the row space of \( A \). Thus, a basis for the row space of \( A \) is \( \{u_1, u_2\} \), where \( u_1 = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 0 & 1 & 1 & -1 \end{bmatrix} \).

e) To use the technique of the row space to find a basis for \( \mathcal{R}(A) \), we perform row operations on \( A^T \) until we reach reduced echelon form. Doing so gives the matrix \( B^T = \begin{bmatrix} 1 & 0 & 6 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \),
which means \( B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix} \). Thus, a basis for \( \mathcal{R}(A) \) is \( \{u_1, u_2\} \), where \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) and
\[
u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.\]