Section 1.2

Definition
An \((m \times n)\) matrix \(B\) is in echelon form if:

1. All rows that consist entirely of zeros are grouped together at the bottom of the matrix.
2. In every nonzero row, the first nonzero entry (counting from left to right) is a 1.
3. If the \((i + 1)\)-st row contains nonzero entries, then the first nonzero entry is in a column to the right of the first nonzero entry in the \(i\)th row.

Definition
A matrix that is in echelon form is in **reduced echelon form** provided that the first nonzero entry in any row is the only nonzero entry in its column.

Remark
Let \([A|b]\) be the augmented matrix for an \((m \times n)\) linear system of equations, and let \([A|b]\) be in reduced echelon form. If the last nonzero row of \([A|b]\) has its leading 1 in the last column, then the system of equations has no solution.

Theorem
Let \(B\) an \((m \times n)\) matrix. There is a unique \((m \times n)\) matrix \(C\) such that:

a) \(C\) is in reduced echelon form and

b) \(C\) is row equivalent to \(B\).

Reduction to Reduced Echelon Form for an \((m \times n)\) Matrix

**Step 1.** Locate the first (left-most) column that contains a nonzero entry

**Step 2.** If necessary, interchange the first row with another row so that the first nonzero column has a nonzero entry in the first row.

**Step 3.** If \(a\) denotes the leading nonzero entry in row one, multiply each entry in row one by \(1/a\). (Thus, the leading nonzero entry in row one is a 1.)

**Step 4.** Add appropriate multiples of row one to each of the remaining rows so that every entry below the leading 1 in row one is a 0.

**Step 5.** Temporarily ignore the first row of this matrix and repeat Steps 1 – 4 on the submatrix that remains. Stop the process when the resulting matrix is in echelon form.

**Step 6.** Having reached echelon form in Step 5, continue on to reduced echelon form as follows: Proceeding upward, add multiples of each nonzero row to the rows above in order to zero all entries above the leading 1.

Solving a system of equations
Given a system of equations:

**Step 1.** Create the augmented matrix for the system.

**Step 2.** Transform the matrix in Step 1 to reduced echelon form.

**Step 3.** Decode the reduced matrix found in Step 2 to obtain its associated system of equations. (This system is equivalent to the original system.)

**Step 4.** By examining the reduced system in Step 3, describe the solution set for the original system.
Problem 1. Consider the matrices below,

a) Either state that the matrix is in the echelon form or use elementary row operations to transform it to echelon form.

b) If the matrix is in echelon form, transform it to reduced echelon form.

\[
\begin{pmatrix}
2 & 0 & 3 & 1 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 2 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & 2 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 2 \\
\end{pmatrix}
\]

Problem 2. Solve the system by transforming the augmented matrix to reduced echelon form.

\[
\begin{align*}
2x_1 & - 3x_2 = 5 \\
-4x_1 & + 6x_2 = -10 \\
\end{align*}
\]

\[
\begin{align*}
x_1 + x_2 - x_5 &= 1 \\
x_2 + 2x_3 + x_4 + 3x_5 &= 1 \\
x_1 - x_3 + x_4 + x_5 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
x_1 + 2x_2 &= 1 \\
2x_1 + 4x_2 &= 2 \\
x_1 - 2x_2 &= -1 \\
\end{align*}
\]

Problem 3. Find all values \( a \) for which the system has no solution.

\[ 2x_1 + 4x_2 = a \]

\[ 3x_1 + 6x_2 = 5 \]

Problem 4. Display all possible configurations for a \((3 \times 3)\) matrix in echelon form.

Problem 5. A certain three-digit number \( N \) equals fifteen times the sum of its digits. If its digits are reversed, the resulting number exceeds \( N \) by 396. The one’s digit is one larger than the sum of the other two. Give a linear system of three equations whose three unknowns are the digits of \( N \). Solve the system and find \( N \).

If \( n \) and \( r \) are positive integers, then there are constants \( a_1, a_2, \ldots, a_{r+1} \) such that

\[ 1^r + 2^r + 3^r + \ldots + n^r = a_1n + a_2n^2 + a_3n^3 + \ldots + a_{r+1}n^{r+1}. \]

Problem 6. Use the equation above to find the formula for the following sum.

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 \]

Homework: Read Section 1.2, do 1, 2, 7, 10, 21, 23, 26, 28, 35, 37, 41, 43, 45, 50, 52, 54, 56.