Section 1.3

Solution possibilities for a Consistent Linear System
Consider the \((m \times n)\) system of linear equations:
\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]
Let \([A|b]\) denote the augmented matrix for the system above. We can use row operations to transform the \([m \times (n+1)]\) matrix \([A|b]\) to a row equivalent matrix \([C|d]\) where \([C|d]\) is in reduced echelon form. Let us consider the following:

- The system represented by the matrix \([C|d]\) is inconsistent if and only if \([C|d]\) has a row of the form \([0,0,\ldots,0,1]\).
- Every variable corresponding to a leading 1 in \([C|d]\) is a dependent variable (leading-one variable).
- Let \(r\) denote the number of nonzero rows in \([C|d]\), then \(r \leq n + 1\).
- Let \(r\) denote the number of nonzero rows in \([C|d]\). If the system represented by \([C|d]\) is consistent, then \(r \leq n\).

Theorem
Let \([C|d]\) be an \([m \times (n+1)]\) matrix in reduced echelon form, where \([C|d]\) represents a consistent system. Let \([C|d]\) have \(r\) nonzero rows. Then \(r \leq n\) and in the solution of the system there are \(n - r\) variables that can be assigned arbitrary values.

Corollary
Consider a \((m \times n)\) system of linear equations. If \(m < n\), then either the system is inconsistent or it has infinitely many solutions.

Homogeneous Systems
The \((m \times n)\) system of linear equations is called a homogeneous system of linear equations:
\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{align*}
\]
A homogeneous system is always consistent, because \(x_1 = x_2 = \cdots = x_n = 0\) is a solution to the system above. This solution is called the trivial solution or zero solution, and any other solution is called a non-trivial solution.

Theorem
A homogeneous \((m \times n)\) system of linear equations always has infinitely many nontrivial solutions when \(m < n\).

Conic Sections
If in the following equation \(ax^2 + bxy + cy^2 + dx + ey + f = 0\) at least one of \(a, b\) or \(c\) is nonzero, the resulting graph is a curve in the \(xy\)-plane. Conic sections include such familiar planes figures as parabola, ellipses, hyperbolas and certain degenerate forms such as points and lines.
Problem 1. Determine all possibilities for the solution set of the system of linear equations described.

a) A homogeneous system of 4 equations in 5 unknowns.

b) A homogeneous system of 4 equations in 3 unknowns.

c) A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1$, $x_2 = 3$, $x_3 = -1$.

d) A system of 3 equations in 4 unknowns that has $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = -3$ as a solution.

Problem 2. Determine whether the given system has nontrivial solutions or only the trivial solution.

\[
\begin{align*}
  x_1 + 2x_2 - x_3 + 2x_4 &= 0 \\
  2x_1 + x_2 + x_3 - x_4 &= 0 \\
  3x_1 - x_2 - 2x_3 + 3x_4 &= 0
\end{align*}
\]

Problem 3. Consider the system of equations,

\[
\begin{align*}
  x_1 + 3x_2 - x_3 &= b_1 \\
  x_1 + 2x_2 &= b_2 \\
  3x_1 + 7x_2 - x_3 &= b_3
\end{align*}
\]

a) Determine conditions on $b_1$, $b_2$, $b_3$ that are necessary and sufficient for the system to be consistent.

b) Give an example of values for $b_1$, $b_2$, $b_3$ such that the linear system is inconsistent.

Problem 4. Find the equation of the conic section through $(-4,0)$, $(-2,-2)$, $(0,3)$, $(1,1)$, and $(4,0)$.

Homework: Read Section 1.3, do 1, 4-7, 9-21 (odd), 29, 31, 32.