Section 3.2

Theorem
If \( x, y \) and \( z \) are vectors in \( \mathbb{R}^n \) and \( a \) and \( b \) are scalars, then the following properties hold:

Closure Properties:
\[
\begin{align*}
\text{a)} & \quad x + y \text{ is in } \mathbb{R}^n. \\
\text{b)} & \quad ax \text{ is in } \mathbb{R}^n
\end{align*}
\]

Properties of addition
\[
\begin{align*}
\text{a)} & \quad x + y = y + x \\
\text{b)} & \quad x + (y + z) = (x + y) + z \\
\text{c)} & \quad \mathbb{R}^n \text{ contains the zero vector, } \theta, \text{ and } x + \theta = x \text{ for all } x \text{ in } \mathbb{R}^n, \\
\text{d)} & \quad \text{For each vector } x \text{ in } \mathbb{R}^n, \text{ there is a vector } -x \text{ in } \mathbb{R}^n \text{ such that } x + (-x) = \theta.
\end{align*}
\]

Properties of scalar multiplication
\[
\begin{align*}
\text{a)} & \quad a(bx) = (ab)x \\
\text{b)} & \quad a(x + y) = ax + by \\
\text{c)} & \quad (a + b)x = ax + bx \\
\text{d)} & \quad 1x = x \text{ for all } x \text{ in } \mathbb{R}^n
\end{align*}
\]

Theorem
A subset \( W \) of \( \mathbb{R}^n \) is a subspace of \( \mathbb{R}^n \) if and only if the following conditions are met:
\[
\begin{align*}
\text{a)} & \quad \text{The zero vector, } \theta, \text{ is in } W. \\
\text{b)} & \quad x + y \text{ is in } W \text{ whenever } x \text{ and } y \text{ are in } W. \\
\text{c)} & \quad ax \text{ is in } W \text{ whenever } x \text{ is in } W \text{ and } a \text{ is any scalar.}
\end{align*}
\]

Problem 1. Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). In each case determine whether \( W \) is a subspace of \( \mathbb{R}^2 \).
\[
\begin{align*}
\text{a)} & \quad W = \{ x : x_1 - x_2 = 2 \} \\
\text{b)} & \quad W = \{ x : x_1 \text{ and } x_2 \text{ are rational numbers } \} \\
\text{c)} & \quad W = \{ x : x_1 x_2 = 0 \}
\end{align*}
\]

Problem 2. Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \). In each case determine whether \( W \) is a subspace of \( \mathbb{R}^3 \).
\[
\begin{align*}
\text{a)} & \quad W = \{ x : x_2 = x_3 + x_1 \} \\
\text{b)} & \quad W = \{ x : x_1^2 = x_1 + x_2 \} \\
\text{c)} & \quad W = \{ x : x_3 = x_2 = 2x_1 \}
\end{align*}
\]
Problem 3. Let \( a \) be a fixed vector in \( \mathbb{R}^3 \), and define \( W \) to be the subset of \( \mathbb{R}^3 \) given by
\[
W = \{ x : a^T x = 0 \}.
\]
Prove that \( W \) is a subspace of \( \mathbb{R}^3 \).

Problem 4. Let \( a \) and \( b \) be fixed vectors in \( \mathbb{R}^3 \), and let \( W \) be the subset of \( \mathbb{R}^3 \) defined by
\[
W = \{ x : a^T x = 0 \text{ and } b^T x = 0 \}.
\]
Prove that \( W \) is a subspace of \( \mathbb{R}^3 \).

Problem 5. If \( U \) and \( V \) are subsets of \( \mathbb{R}^n \), then the set \( U + V \) is defined by
\[
U + V = \{ x : x = u + v, u \in U \text{ and } v \in V \}.
\]
Prove that if \( U \) and \( V \) are subspaces of \( \mathbb{R}^n \), then \( U + V \) is a subspace of \( \mathbb{R}^n \).

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Homework: Read Section 3.2, do 1, 3, 7, 9, 11, 15, 19, 23, 24, 27, 29, 31