Section 3.4

Spanning Sets
Let $W$ be a subspace of $\mathbb{R}^n$, and let $S = \{w_1, \ldots, w_m\}$ be a subset of $W$. We say that $S$ is a spanning set for $W$, or simply that $S$ spans $W$, if every vector $w$ in $W$ can be expressed as a linear combination of vectors in $S$:

$$w = a_1w_1 + a_2w_2 + \cdots + a_mw_m.$$ 

Definition
Let $W$ be a nonzero subspace of $\mathbb{R}^n$. A basis for $W$ is a linearly independent spanning set for $W$.

Uniqueness of Representation
Let $B = \{v_1, v_2, \ldots, v_p\}$ be a basis for a subspace $W$ of $\mathbb{R}^n$, and let $x$ be a vector in $W$. Because $B$ is a spanning set, we know that there are scalars $a_1, a_2, \ldots, a_p$ such that

$$x = a_1v_1 + a_2v_2 + \cdots + a_pv_p.$$ 

Then the representation of $x$ is unique.

Finding a Basis
a) A spanning set $S = \{v_1, \ldots, v_m\}$ for a subspace $W$ is given.

b) Solve the vector equation $x_1v_1 + \cdots + x_mv_m = \theta$.

c) If the equation has only the trivial solution $x_1 = \cdots = x_m = 0$, then $S$ is a linearly independent set and hence is a basis for $W$.

d) If the equation has a nontrivial solutions, then there are unconstrained variables. For each $x_j$ that is designated as an unconstrained variable, delete the vector $v_j$ from the set $S$. The remaining vectors constitute a basis for $W$.

Theorem
If a nonzero matrix $A$ is row equivalent to the matrix $B$ in echelon form, then the nonzero rows of $B$ form a basis for the row space of $A$.

Finding a Basis Using the Row Space
a) A spanning set $S = \{v_1, \ldots, v_m\}$ for a subspace $W$ of $\mathbb{R}^n$ is given.

b) Let $V$ be the $n \times m$ matrix $V = [v_1, \ldots, v_m]$. Use elementary row operations to transform $V^T$ to a matrix $B^T$ in echelon form.

c) The nonzero columns of $B$ are a basis for $W$. 
Problem 1. Let $W$ be the subspace of $\mathbb{R}^4$ consisting of vectors of the form

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \]

Find a basis of $W$ when the components of $\mathbf{x}$ satisfy the given conditions.

\begin{align*}
&x_1 + x_2 - x_3 + x_4 = 0 \\
&x_2 - 2x_3 - x_4 = 0
\end{align*}

Determine if $\mathbf{x}$ is in $W$. If $\mathbf{x}$ is in $W$, then express $\mathbf{x}$ as a linear combination of the basis vectors.

\[ \mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 3 \\ 2 \end{bmatrix} \]

Problem 2. For the following matrices.

a) Find a matrix $B$ in reduced echelon form such that $B$ is row equivalent to the given matrix $A$.

b) Find a basis for the null space of $A$.

c) Find a basis for the range of $A$ that consists of columns of $A$. For each column, $A_j$, of $A$ that does not appear in the basis, express $A_j$ as a linear combination of the basis vectors.

d) Exhibit a basis for the row space of $A$.

e) Use the technique of the row space to find a basis for the range of $A$.

\[ A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \]

Problem 3. In the following exercises for the given set $S$:

a) Find a subset $S$ that is a basis for $\text{Sp}(S)$ using the first technique to find a basis.

b) Find a basis for $\text{Sp}(S)$ using the second technique to find a basis.

\[ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \]

\[ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \right\} \]

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Homework: Read Section 3.4, do 1, 3, 7, 9, 11, 15, 17, 23, 25, 33.