Theorem
Let \( W \) be a subspace of \( \mathbb{R}^n \), and let \( B = \{ w_1, w_2, \ldots, w_p \} \) be a spanning set for \( W \) containing \( p \) vectors. Then any set of \( p + 1 \) or more vectors in \( W \) is linearly dependent.

Corollary
Let \( W \) be the subspace of \( \mathbb{R}^n \), and let \( B = \{ w_1, w_2, \ldots, w_p \} \) be a basis for \( W \) containing \( p \) vectors. Then every basis for \( W \) contains \( p \) vectors.

Definition
Let \( W \) be a subspace of \( \mathbb{R}^n \). If \( W \) has a basis \( B = \{ w_1, w_2, \ldots, w_p \} \) of \( p \) vectors, then we say that \( W \) is a subspace of dimension \( p \), and we write \( \text{dim}(W) = p \).

Theorem
Let \( W \) be a subspace of \( \mathbb{R}^n \) with \( \text{dim}(W) = p \).

a) Any set of \( p + 1 \) or more vectors in \( W \) is linearly dependent.

b) Any set of fewer than \( p \) vectors in \( W \) does not span \( W \).

c) Any set of \( p \) linearly independent vectors in \( W \) is a basis for \( W \).

d) Any set of \( p \) vectors that spans \( W \) is a basis for \( W \).

Rank of a Matrix
For an \( m \times n \) matrix, the dimension of the null space is called the nullity of \( A \), and the dimension of the range of \( A \) is called the rank of \( A \).

Theorem
If \( A \) is an \( m \times n \) matrix, then the rank of \( A \) is equal to the rank of \( A^T \).

Corollary
If \( A \) is an \( m \times n \) matrix, then the row space and the column space of \( A \) have the same dimension.

Remark
If \( A \) is an \( m \times n \) matrix, then \( n = \text{rank}(A) + \text{nullity}(A) \).

Theorem
An \( m \times n \) system of linear equations, \( Ax = b \), is consistent if and only if, \( \text{rank}(A) = \text{rank}([A|b]) \).

Theorem
An \( n \times n \) matrix \( A \) is nonsingular if and only if the rank of \( A \) is \( n \).

Problem 1. Determine whether the given set is a basis for \( \mathbb{R}^3 \).

a) \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \)

b) \( S = \{ \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \)

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}.
\]
Problem 2. $W$ is a subspace of $\mathbb{R}^4$ consisting of vectors of the form $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Determine $\dim(W)$ when the components of $x$ satisfy the given conditions.

a) $x_1 + x_3 - 2x_4 = 0$
   $x_2 + 2x_3 - 3x_4 = 0$

b) $x_1 - x_2 = 0$
   $x_2 - 2x_3 = 0$
   $x_3 - x_4 = 0$

Problem 3. Find a basis for $N(A)$ and give the nullity and the rank of $A$.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -5 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \end{bmatrix}$$

Problem 4. Find a basis for $R(A)$ and give the nullity and the rank of $A$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

Problem 5. Let $W$ be a subspace, and let $S$ be a spanning set for $W$. Find a basis for $W$, and calculate $\dim(W)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \\ 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$$

Problem 6. Let $W$ be a subspace of $\mathbb{R}^4$ defined by $W = \{ x : v^T x = 0 \}$. Calculate $\dim(W)$, where

$$v = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$$

Problem 7. For each of the following, determine the largest possible value for the rank of $A$ and the smallest possible value for the nullity of $A$.

a) $A$ is $3 \times 3$

b) $A$ is $3 \times 4$

c) $A$ is $5 \times 4$

Homework: Read Section 3.5, do 3, 5, 7, 9, 11, 13, 17, 23, 25, 27(a), 29, 31.