Orthogonal Vectors
If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^n \), we say that \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if \( \mathbf{u}^T \mathbf{v} = 0 \).

Orthogonal Set
Let \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) be a set of vectors in \( \mathbb{R}^n \). The set \( S \) is said to be an orthogonal set if each pair of distinct vectors from \( S \) is orthogonal; that is, \( \mathbf{u}_i^T \mathbf{u}_j = 0 \) when \( i \neq j \).

Theorem
Let \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) be a set of nonzero vectors in \( \mathbb{R}^n \). If \( S \) is an orthogonal set of vectors, then \( S \) is a linearly independent set of vectors.

Orthogonal Basis
Let \( \mathcal{W} \) be a subspace of \( \mathbb{R}^n \), and let \( B = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) be a basis for \( \mathcal{W} \). If \( B \) is an orthogonal set of vectors, then \( B \) is called an orthogonal basis for \( \mathcal{W} \).

Furthermore, if \( \| \mathbf{u}_i \| = 1 \) for \( 1 \leq i \leq p \), then \( B \) is said to be an orthonormal basis for \( \mathcal{W} \).

Orthonormal Bases
If \( B = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal set, then \( C = \{ a_1 \mathbf{u}_1, a_2 \mathbf{u}_2, \ldots, a_p \mathbf{u}_p \} \) is also an orthogonal set for any scalars \( a_1, a_2, \ldots, a_p \). If \( B \) contains only nonzero vectors and if we define the scalars \( a_i \) by
\[
a_i = \frac{1}{\sqrt{\mathbf{u}_i^T \mathbf{u}_i}},
\]
then \( C \) is an orthonormal set. We can convert an orthogonal set of nonzero vectors into an orthonormal set by dividing each vector by its length.

Coordinates
Let \( \mathcal{W} \) be a subspace of \( \mathbb{R}^n \), and let \( B = \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p \} \) be an orthogonal basis for \( \mathcal{W} \). If \( \mathbf{v} \) is any vector in \( \mathcal{W} \), then \( \mathbf{v} \) can be expressed uniquely in the form
\[
\mathbf{v} = a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2 + \cdots + a_p \mathbf{w}_p,
\]
where
\[
a_i = \frac{\mathbf{w}_i^T \mathbf{v}}{\mathbf{w}_i^T \mathbf{w}_i}, \quad 1 \leq i \leq p.
\]

Theorem
Let \( \mathcal{W} \) be a \( p \)-dimensional subspace of \( \mathbb{R}^n \), and let \( \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p \} \) be any basis for \( \mathcal{W} \). Then the set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal basis for \( \mathcal{W} \), where
\[
\mathbf{u}_1 = \mathbf{v}_1,
\]
\[
\mathbf{u}_2 = \mathbf{w}_2 - \frac{\mathbf{u}_1^T \mathbf{w}_2}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1
\]
\[
\mathbf{u}_3 = \mathbf{w}_3 - \frac{\mathbf{u}_1^T \mathbf{w}_3}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{u}_2^T \mathbf{w}_3}{\mathbf{u}_2^T \mathbf{u}_2} \mathbf{u}_2,
\]
in general,
\[
\mathbf{u}_i = \mathbf{w}_i - \sum_{k=1}^{i-1} \frac{\mathbf{u}_k^T \mathbf{w}_i}{\mathbf{u}_k^T \mathbf{u}_k} \mathbf{u}_k, \quad 2 \leq i \leq p.
\]
Problem 1. Verify that \{u_1, u_2, u_3\} is an orthogonal set for the given vectors.

\[
\begin{align*}
  u_1 &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.
\end{align*}
\]

Problem 2. Find the values \(a, b, c\) such that \{u_1, u_2, u_3\} is an orthogonal set.

\[
\begin{align*}
  u_1 &= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} a \\ 1 \\ -1 \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} b \\ 3 \\ c \end{bmatrix}.
\end{align*}
\]

Problem 3. Express the given vector \(v\) in terms of the orthogonal basis \(B = \{u_1, u_2, u_3\}\).

\[
\begin{align*}
  v &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \\
  u_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}.
\end{align*}
\]

Problem 4. Use the Gram-Schmidt process to generate an orthogonal set from the given linearly independent vectors.

a) \[
\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ -5 \\ 5 \end{bmatrix},
\]

b) \[
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}
\]

Problem 5. Find a basis for the null space and the range of the given matrix. Then use Gram-Schmidt to obtain orthogonal bases.

\[
A = \begin{bmatrix} 1 & 3 & 10 & 11 & 9 \\ -1 & 2 & 5 & 4 & 1 \\ 2 & -1 & -1 & 1 & 4 \end{bmatrix}
\]

Problem 6. The Cauchy-Schwartz inequality. Let \(x\) and \(y\) be vectors in \(\mathbb{R}^n\). Prove that \(|x^T y| \leq \|x\| \|y\|\).

Problem 7. The Triangle Inequality. Let \(x\) and \(y\) be vectors in \(\mathbb{R}^n\). Prove that \(\|x + y\| \leq \|x\| + \|y\|\).

Problem 8. Let \(x\) and \(y\) be vectors in \(\mathbb{R}^n\). Prove that \(||x|| - ||y||\) \leq \(||x - y||\).

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Homework: Read Section 3.6, do 3, 7, 11, 15, 17, 19, 23.