Determinant for $2 \times 2$ Matrices
Let $A$ be the $2 \times 2$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinant of $A$, denoted by $\det(A)$, is the number

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$ 

Determinant of $3 \times 3$ Matrices
Let $A$ be the $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of $A$ is the number $\det(A)$, where

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{12} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Minor
Let $A = (a_{ij})$ be an $n \times n$ matrix. The $(n-1) \times (n-1)$ matrix that results from removing the $r$th row and $s$th column from $A$ is called a minor of a matrix of $A$ and is designated by $M_{rs}$.

Cofactor
The numbers $A_{ij}$ defined by

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

are known as a cofactor, where $M_{ij}$ is the minor of the $ij$-th entry $a_{ij}$.

The Determinant of an $n \times n$ matrix
Let $A = (a_{ij})$ be an $n \times n$ matrix. The determinant of $A$ is the number $\det(A)$, where

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \ldots + (-1)^{n+1}a_{1n}\det(M_{1n}) = \sum_{j=1}^{n} (-1)^{j+1}a_{1j}\det(M_{1j}).$$

Theorem
Let $A = (a_{ij})$ be an $n \times n$ matrix with minor matrices $M_{ij}$ and cofactors $A_{ij} = (-1)^{i+j}\det(M_{ij})$. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} \text{ (i-th row expansion)}$$
$$\det(A) = \sum_{i=1}^{n} a_{ij}A_{ij} \text{ (j-th column expansion)}$$

Theorem
Let $A$ and $B$ be $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$
Theorem
Let $A$ be an $n \times n$ matrix. Then

$$A \text{ is singular if and only if } \det(A) = 0.$$  

Theorem
Let $T = (t_{ij})$ be an $n \times n$ triangular matrix. Then

$$\det(T) = t_{11}t_{22}\ldots t_{nn}.$$  

Problem 1. Calculate the determinant of the given matrix. State whether the matrix is singular or nonsingular.

\[ A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}. \]

Problem 2. Let $A$ and $B$ be $n \times n$ matrices. Give a proof of each of the following.

(a) If either $A$ or $B$ is singular, then $AB$ is singular.

(b) If $AB$ is singular, then either $A$ or $B$ is singular.

Problem 3. Suppose that $A$ is an $n \times n$ nonsingular matrix. Show that $\det(A^{-1}) = 1/\det(A)$.

Problem 4. Evaluate the given determinant, where $A$ and $B$ are $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = 5$.

(a) $\det(A^2B)$.

(b) $\det(AB^{-1}A^{-1}B^2)$.

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Homework: Read Section 4.2, do 9, 13, 17, 19, 21, 27, 29.