Section 4.5

Eigenspaces and Geometric Multiplicity
Let $A$ be an $n \times n$ matrix. If $\lambda$ is an eigenvalue of $A$, then:

a) The null space of $A - \lambda I$ is denoted by $E_\lambda$ and is called the eigenspace of $\lambda$.

b) The dimension of $E_\lambda$ is called the geometric multiplicity of $\lambda$.

Defective Matrices
Let $A$ be an $n \times n$ matrix. If there is an eigenvalue $\lambda$ of $A$ such that the geometric multiplicity of $\lambda$ is less than the algebraic multiplicity of $\lambda$, then $A$ is called a defective matrix.

Theorem
Let $u_1, u_2, \ldots, u_k$ be eigenvectors of an $n \times n$ matrix $A$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. That is,

$$Au_j = \lambda_j u_j \quad \text{for } j = 1, 2, \ldots, k; \ k \leq n$$

$$\lambda_i \neq \lambda_j \text{ for } i \neq j; \ 1 \leq i, j \leq k.$$ 

Then $\{u_1, u_2, \ldots, u_k\}$ is a linearly independent set.

Corollary
Let $A$ be an $n \times n$ matrix. If $A$ has $n$ distinct eigenvalues, then $A$ has a set of linearly independent eigenvectors.

Problem 1. Find the eigenvalues and the eigenvectors for the given matrix, and find a basis for the eigenspace $E_\lambda$. Determine the algebraic and geometric multiplicity of $\lambda$.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 6 & 2 \\ 0 & 5 & -6 \\ 1 & 0 & -2 \end{bmatrix}, \quad p(t) = -(t + 4)(t - 3)^2$$

$$C = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad p(t) = (t + 2)(t - 2)^3.$$
Problem 2. If a vector $x$ is a linear combination of eigenvectors of a matrix $A$, then it is easy to calculate the product $y = A^kx$ for any positive integer $k$. For instance, suppose that $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$, where $u_1$ and $u_2$ are nonzero vectors. If $x = a_1 u_1 + a_2 u_2$, then $y = A^k x = A^k(a_1 u_1 + a_2 u_2) = a_1 A^k u_1 + a_2 A^k u_2 = a_1 (\lambda_1)^k u_1 + a_2 (\lambda_2)^k u_2$. Find $A^{10} x$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -2 \\ 0 & 6 & -2 \end{bmatrix} \text{ and } x = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

Problem 3. Let $P$ be an idempotent matrix ($P^2 = P$). Show that the only eigenvalues of $P$ are $\lambda = 0$ and $\lambda = 1$.

Problem 4. Let $u$ be a vector in $\mathbb{R}^n$ such that $u^T u = 1$. Show that the $n \times n$ matrix $P = uu^T$ is an idempotent matrix.

Problem 5. Verify that if $Q$ is idempotent, then so is $I - Q$. Also verify that $(I - 2Q)^{-1} = I - 2Q$.

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Homework: Read Section 4.5, do 1, 5, 9, 11, 13, 17, 19, 21, 23, 25.