Math 2313, Final

Name ___________________________

1. Find a vector that is parallel to the plane $2x - 3y + z = 10$ and perpendicular to the line $x = -1 + 2t, y = 3t, z = 2 - t$.

answer: $(2, -3, 1) \times (2, 3, -1) = (0, 4, 12)$

2. Consider the function $f(x, y, z) = \ln(xy) + e^{xz}$:

a. Find the gradient of $f$ at $(1, 1, 0)$.

answer: $<1, 1, 1>$

b. Find the directional derivative of $f$ at $(1, 1, 0)$ in the direction of the vector $<3, 4, 12>$.

answer: $19/13$

c. Find the equation of the tangent plane to the surface $f(x, y, z) = 1$, at $(1, 1, 0)$.
answer: $x + y + z = 2$

3. a. Consider the curve defined by $r(t) = \langle \cos(\pi t), 4t, \sin(\pi t) \rangle$. Find the equations of the tangent line to this curve at the point $(-1, 4, 0)$.

answer: $x = -1, y = 4 + 4t, z = -\pi t$

b. Find the length of this curve between the points $(1, 0, 0)$ and $(-1, 4, 0)$.

answer: $\sqrt{\pi^2 + 16}$

4. Express $(\frac{\partial u}{\partial r})^2 + (\frac{1}{r} \frac{\partial u}{\partial \theta})^2$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, using the chain rule, and simplify as much as possible. (Hint: $x = r \cos(\theta), y = r \sin(\theta)$)

answer: $u_x^2 + u_y^2$
5. a. Evaluate $\int_0^1 \int_y^1 y \, dx \, dy$.

answer: $1/6$

b. Reverse the order of integration and evaluate.

answer: $\int_0^1 \int_0^x y \, dy \, dx = 1/6$

c. Rewrite using polar coordinates and evaluate. (Hint: $x = 1$ is $r \cos(\theta) = 1$.)

answer: $\int_{\pi/4}^{\pi/2} \int_0^{1/\cos(\theta)} r \sin(\theta) r \, dr \, d\theta = 1/6$

6. a. The region defined by $x > 0, y > 0, z > 0, x+y+z < 1$ has density $\rho(x,y,z)$. Set up the integrals required to find the x-coordinate $\bar{x}$ of the center of gravity $(\bar{x}, \bar{y}, \bar{z})$.

answer: $M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \rho(x,y,z) \, dz \, dy \, dx$
\[ \bar{x} = \frac{1}{M} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \rho(x, y, z) dz \, dy \, dx \]

b. Set up the integral to find the surface area of the slanted top of the tetrahedron of part (a), and evaluate it.

Answer: \( \int_0^1 \int_0^{1-x} \sqrt{3} \, dy \, dx = \sqrt{3}/2 \)