1.5 Infinite Limits

Definition of Infinite Limits – Let \( f \) be a function that is defined at every real number in some open interval containing \( c \) (except possibly at \( c \) itself). The statement \( \lim_{x \to c} f(x) = \infty \) means that for each \( M > 0 \) there exists a \( \delta > 0 \) such that \( f(x) > M \) whenever \( 0 < |x - c| < \delta \). Similarly, the statement \( \lim_{x \to c} f(x) = -\infty \) means that for each \( N < 0 \) there exists \( \delta > 0 \) such that \( f(x) < N \) whenever \( 0 < |x - c| < \delta \). To define the infinite limit from the left, replace \( 0 < |x - c| < \delta \) by \( c - \delta < x < c \). To define the infinite limit from the right, replace \( 0 < |x - c| < \delta \) by \( c < x < c + \delta \).

Note 1: Having a limit equal to infinity does NOT mean that the limit exists. In fact, it means the limit is unbounded and therefore does not exist.

Note 2: In WebAssign, if the limit is \( \infty \) you should enter that for your answer and not DNE. There does not seem to be a great consistency in this however.

Definition of Vertical Asymptote – If \( f(x) \) approaches infinity (positive or negative) as \( x \) approaches \( c \) from the right or the left, then the line \( x = c \) is a vertical asymptote of the graph of \( f(x) \).

Theorem on Vertical Asymptotes – Let \( f \) and \( g \) be continuous on an open interval containing \( c \). If \( f(c) \neq 0, \quad g(c) = 0 \), and there exists and open interval containing \( c \) such that \( g(x) \neq 0 \) for all \( x \neq c \) in the interval, then the graph of the function given by \( h(x) = \frac{f(x)}{g(x)} \) has a vertical asymptote at \( x = c \).

Examples: Find the vertical asymptotes, if any.

1. \( f(x) = \frac{-4x}{x^2 + 4} \quad \text{ numerator is zero when } x = 0 \)
   \( \quad \text{ denominator is never zero for real } x \)
   Therefore, no vertical asymptotes.
2. \( h(s) = \frac{2s-3}{s^2-25} \)  
   \( \text{numerator is zero at } s = \frac{3}{2} \)  
   \( \text{denom. is zero at } s = \pm 5 \)  
   \( \text{Vertical asymptotes are } s = -5 \text{ and } s = +5 \)  
   \( \text{Note: vertical asymptotes are always equations of lines} \) 

3. \( g(x) = \frac{x^3+1}{x+1} = \frac{(x+1)(x^2+x+1)}{x+1} \)  
   \( \text{Both numerator and denominator are zero at } x = -1. \text{ This is not a vertical asymptote.} \) 

4. \( h(t) = \frac{t^2-2t}{t^4-16} = \frac{t(t-2)}{(t-2)(t+2)(t^2+4)} \)  
   \( t = 2 \) is not a vertical asymptote  
   as it is a removable discontinuity.  
   However, \( t = -2 \) is a vertical asymptote. 

5. \( f(x) = \sec(\pi x) = \frac{1}{\cos(\pi x)} \)  
   The numerator is never zero.  
   The denominator is zero when \( \pi x = \frac{n\pi}{2} \), where \( n \) is an arbitrary integer.  
   This means \( f(x) = \sec(\pi x) \) has vertical asymptotes at \( x = \frac{1}{2} + n \), for all integers \( n \). \( \text{Solve } \pi x = \frac{n\pi}{2} \text{ for } x \).

Examples: Find the limit, if it exists.

1. \( \lim_{x \to 1^+} \frac{2+x}{1-x} \)  
   \( \text{Tables are great if you don't have a graph.} \) 
   \[
   \begin{array}{c|c|c|c}
   x & 1.001 & 1.01 & 1.1 \\
   \hline
   f(x) & -3.001 & -3.01 & -3.1 \\
   \end{array}
   \]
   \( f(x) \) gets large and negative as \( x \to 1^+ \) 
   \( \text{so } \lim_{x \to 1^+} \frac{2+x}{1-x} = -\infty \)
Example: A patrol car is parked 50 feet from a long warehouse. The revolving light on top of the car turns at a rate of \( \frac{\pi}{2} \) revolution per second. The rate at which the light beam moves along the wall is \( r = 50\pi \sec^2 \theta \text{ ft/sec} \).

a) Find the rate \( r \) when \( \theta \) is \( \pi/6 \).

\[
r = 50\pi \sec^2 \left( \frac{\pi}{6} \right) = 50\pi \left( \frac{1}{\sqrt{3}} \right)^2 = \frac{50\pi}{3} = \frac{2 \cos \theta}{3} \approx 209.4 \text{ ft/sec}
\]

As a comparison, the exact value is \( \frac{2 \pi}{3} \).

b) Find the rate \( r \) when \( \theta \) is \( \pi/3 \).

\[
r = 50\pi \sec^2 \left( \frac{\pi}{3} \right) = 50\pi \cdot \frac{1}{(\sqrt{3})^2} = 200\pi \approx 628.3 \text{ ft/sec}
\]

c) Find the limit of \( r \) as \( \theta \to (\pi/2)^- \).

We know that \( r \) will be \( +\infty \) at \( \pi/2 \), but we need to determine which one it is. Notice in parts (a) and (b) that \( r \) is increasing in a positive direction. This leads to the conclusion \( \lim_{x \to \pi/2^-} r = +\infty \).
Example: A 25 foot ladder is leaning against a house. If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of \( r = \frac{25}{\sqrt{625-x^2}} \) ft/sec where \( x \) is the distance between the base of the ladder and the house.

a) Find the rate when \( x \) is 7 feet.

\[
\sqrt{x} = \frac{2.5}{\sqrt{625-49}} = \frac{2.5}{\sqrt{576}} = \frac{25}{20} \approx 1.04 \text{ ft/sec}
\]

b) Find the rate when \( x \) is 15 feet.

\[
\sqrt{x} = \frac{2.5}{\sqrt{625-225}} = \frac{2.5}{\sqrt{400}} = \frac{25}{20} = 1.25 \text{ ft/sec}
\]

Exact, no rounding used

c) Find the limit of \( r \) as \( x \to 25^- \) Let's try a table... although initial guess from the pattern is +\( \infty \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>24.1</th>
<th>24.5</th>
<th>24.9</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>3.76</td>
<td>5.03</td>
<td>11.19</td>
<td>25</td>
</tr>
</tbody>
</table>

Yes, it seems reasonable as the closer we get to 25 from the left, the higher the rate. \( \lim_{x \to 25^-} r = +\infty \)

Properties of Infinite Limits – Let \( c \) and \( L \) be real numbers and let \( f \) and \( g \) be functions such that \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = L \).

1. Sum or difference: \( \lim_{x \to c} [f(x) \pm g(x)] = \infty \)

2. Product: \( \lim_{x \to c} [f(x) g(x)] = \infty \), \( L > 0 \)

\( \lim_{x \to c} [f(x) g(x)] = -\infty \), \( L < 0 \)

3. Quotient: \( \lim_{x \to c} \frac{g(x)}{f(x)} = 0 \)

Similar properties hold for one-sided limits and for functions for which the limit of \( f(x) \) as \( x \) approaches \( c \) is \(-\infty \).