Chapter Two: Differentiation

2.1 The Derivative and the Tangent Line Problem

The difference quotient is introduced in pre-calculus as a rate of change. This will be the basis of the definition of derivatives.

Definition of Tangent Line with Slope $m$ – If $f$ is defined on an open interval containing $c$, and if the limit

$$
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m
$$

exists, then the line passing through $(c, f(c))$ with slope $m$ is the tangent line to the graph of $f$ at the point $(c, f(c))$.

The slope of the tangent line is also called the slope of the graph.

Examples: Find the slope of the tangent line to the graph of the function at the given point.

1. \( f(x) = \frac{3}{2}x + 1 \), \((-2, -2)\) 
   \( c = -2 \)

   \( f(-2+\Delta x) = \frac{3}{2}(-2+\Delta x)+1 \)
   \( = -3 + \frac{3}{2}\Delta x + 1 \)
   \( = -2 + \frac{3}{2}\Delta x \)

   \( f(-2) = \frac{3}{2}(-2)+1 = -3+1 = -2 \)

   \( m = \lim_{\Delta x \to 0} \frac{-2 + \frac{3}{2}\Delta x - (-2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{3}{2}\Delta x}{\Delta x} = \frac{3}{2} \)

   The slope of the tangent line to the graph of \( f(x) = \frac{3}{2}x+1 \) at \( c = -2 \) is \( \frac{3}{2} \).

Notice that the slope of any linear function is going to also be the slope of the tangent.
2. \( g(x) = 6 - x^2 \), \((1,5)\)
\[
g'(1) = 6 - (1)^2 = 6 - 1 = 5
\]
\[
m = \lim_{{\Delta x \to 0}} \frac{5 - 2\Delta x - (\Delta x)^2 - 5}{\Delta x} = \lim_{{\Delta x \to 0}} \frac{-2\Delta x - (\Delta x)^2}{\Delta x} = \lim_{{\Delta x \to 0}} \frac{\Delta x (-2 - \Delta x)}{\Delta x} = \lim_{{\Delta x \to 0}} (-2 - \Delta x) = \frac{-2}{0} = -0 = -2
\]

3. \( h(t) = t^2 + 3 \), \((-2,7)\)
\[
h'(-2+\Delta t) = (-2+\Delta t)^2 + 3 = (4 - 4\Delta t + (\Delta t)^2) + 3 = 7 - 4\Delta t + (\Delta t)^2
\]
\[
h'(-2) = (-2)^2 + 3 = 4 + 3 = 7
\]
\[
m = \lim_{{\Delta t \to 0}} \frac{7 - 4\Delta t + (\Delta t)^2 - 7}{\Delta t} = \lim_{{\Delta t \to 0}} \frac{-4\Delta t + (\Delta t)^2}{\Delta t} = \lim_{{\Delta t \to 0}} \frac{\Delta t (-4 + \Delta t)}{\Delta t} = \lim_{{\Delta t \to 0}} (-4 + \Delta t) = -4
\]

Definition of the Derivative of a Function – The derivative of \( f \) at \( x \) is given by

\[
f'(x) = \lim_{{\Delta x \to 0}} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]
provided the limits exists. For all \( x \) for which this limit exists, \( f' \) is a function of \( x \).

Notation – The following are equivalent: \( f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y] \)
Examples: Find the derivative by the limit process.

1. \( f(x) = 3x + 2 \)
   \[
   f(x + \Delta x) = 3(x + \Delta x) + 2 = 3x + 3\Delta x + 2
   
   \]
   \[
   f(x + \Delta x) - f(x) = \frac{3\Delta x}{3 \Delta x}
   
   \text{That is, } f'(x) = 3.
   
2. \( g(x) = 2 - x^2 \)
   \[
   g(x + \Delta x) = 2 - (x + \Delta x)^2
   
   = 2 - (x^2 + 2x\Delta x + (\Delta x)^2)
   
   = 2 - x^2 - 2x\Delta x - (\Delta x)^2
   
   g(x + \Delta x) - g(x) = -2x\Delta x - (\Delta x)^2
   
   g'(x) = \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x}
   
   = \lim_{\Delta x \to 0} \frac{-2x - (\Delta x)}{\Delta x}
   
   = -2x
   
3. \( f(x) = \frac{4}{\sqrt{x}} \)
   \[
   f(x + \Delta x) = \frac{4}{\sqrt{x + \Delta x}}
   
   \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\sqrt{x + \Delta x} - \sqrt{x}}
   
   \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\sqrt{x + \Delta x} - \sqrt{x}}
   
   \text{this needs work}
   
   \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\sqrt{x + \Delta x} - \sqrt{x}}
   
   \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\sqrt{x + \Delta x} - \sqrt{x}}
   
   \text{this still isn’t enough because…}
   
   \]
   \[
   f'(x) = \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x}
   
   = \lim_{\Delta x \to 0} \frac{-16\Delta x}{\sqrt{x + \Delta x} \left(4\sqrt{x} + 4\sqrt{x + \Delta x}\right)}
   
   = -\frac{16}{8\sqrt{x}}
   
   And finally, } \frac{f'(x)}{x^\sqrt{x}} = \frac{-2}{\sqrt{x}}
Examples: Find an equation of the tangent line to the graph of \( f \) at the given point.

1. \( f(x) = x^2 + 3x + 4, \ (-2, 2) \)
   \[ f(c) = f(-2) = -2^2 + 3(-2) + 4 = 0 \]
   \[ f'(c) = f'(-2) = 2(-2) + 3 = -4 + 3 = -1 \]
   \[ y - 2 = -1(x + 2) \]
   \[ y = -x + 1 \]

   \[ f(x + \Delta x) = (x + \Delta x)^2 + 3(x + \Delta x) + 4 \]
   \[ = x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 4 \]
   \[ \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x + 3 \]

2. \( f(x) = \sqrt{x - 1}, \ (5, 2) \)
   \[ f(5 + \Delta x) = \sqrt{5 + \Delta x - 1} = \sqrt{4 + \Delta x} \]
   \[ f(5) = \sqrt{5 - 1} = \sqrt{4} = 2 \]
   \[ f(5 + \Delta x) - f(5) = \sqrt{4 + \Delta x} - 2 \]
   \[ \frac{f(5 + \Delta x) - f(5)}{\Delta x} = \frac{\sqrt{4 + \Delta x} - 2}{\Delta x} \]
   \[ = \frac{4 + \Delta x - 4}{(\sqrt{4 + \Delta x} + 2)\Delta x} \]
   \[ = \frac{\Delta x}{(\sqrt{4 + \Delta x} + 2)\Delta x} \]
   \[ = \frac{1}{\sqrt{4 + \Delta x} + 2} \]
   \[ m = \lim_{\Delta x \to 0} \frac{1}{\sqrt{4 + \Delta x} + 2} \]
   \[ y - 2 = \frac{1}{4} (x - 5) \]
   \[ y = \frac{1}{4} x + \frac{3}{4} \]

3. \( f(x) = \frac{1}{x + 1}, \ (0, 1) \)
   \[ f(0 + \Delta x) = \frac{1}{\Delta x + 1} \]
   \[ f(0) = 1 \]
   \[ f(0 + \Delta x) - f(0) = \frac{1}{\Delta x + 1} - 1 \]
   \[ = \frac{1}{\Delta x + 1} - \frac{\Delta x + 1}{\Delta x + 1} \]
   \[ = -\frac{\Delta x}{\Delta x + 1} \]
   \[ y - 1 = -1(x - 0) \]
   \[ y = -x + 1 \]
Example: Find an equation of the line that is tangent to \( f(x) = x^3 + 2 \) and is parallel to \( 3x - y - 4 = 0 \).

\[
\begin{align*}
\frac{f(x + \Delta x) - f(x)}{\Delta x} &= (x + \Delta x)^3 + 2 \\
&= x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2 \\
\end{align*}
\]

\[
\frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2 + 3x \Delta x + (\Delta x)^2
\]

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2
\]

If \( f'(x) = 3x^2 \), then \( f'(1) = 3 \) when \( 3x^2 = 3 \) or \( x = \pm 1 \).

This gives two tangent lines, one through \((-1, 1)\) and one at \((1, 3)\).

Alternative Definition of the Derivative – The derivative of \( f \) at \( c \) is

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

provided this limit exists. Notice that this quotient is just the formula for the slope of a line between two points and the limit is what makes it work for nonlinear functions.

Examples: Use the alternative form of the derivative to find the derivative at \( x = c \), if it exists.

1. \( f(x) = x(x - 1) \), \( c = 1 \)

\[
f'(1) = \lim_{x \to 1} \frac{x(x - 1) - 1(1 - 1)}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{x - 1} = \lim_{x \to 1} x = 1
\]

So \( f'(1) = 1 \).
2. \( f(x) = \frac{2}{x}, c = 5 \)

\[
\lim_{x \to 5} f'(x) = \lim_{x \to 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5} = \frac{-2}{5(x-5)} = \frac{-2}{25}
\]

\( f'(c) = \frac{f'(5)}{f(5)} = \frac{-2}{25} \)