Chapter Four: Integration

4.1 Antiderivatives and Indefinite Integration

Definition of Antiderivative – A function $F$ is an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Representation of Antiderivatives – If $F$ is an antiderivative of $f$ on an interval $I$, then $G$ is an antiderivative of $f$ on the interval $I$ if and only if $G$ is of the form $G(x) = F(x) + C$, for all $x$ in $I$ where $C$ is a constant.

This tells us that all antiderivatives of a function differ only by a constant $C$.

Examples: Find an antiderivative and then find the general antiderivative.

1. $y = 3$
   
   A possible antiderivative is $y = 3x$. In general, the antiderivative is $y = 3x + C$.

2. $f(x) = 2x$
   
   A possible antiderivative is $F(x) = x^2$. In general, $F(x) = x^2 + C$.

3. $f(x) = 5x^4$
   
   A possible antiderivative is $F(x) = x^5 + 7$. In general, $F(x) = x^5 + C$.

Notation: If we take the differential form of a derivative, $\frac{dy}{dx} = f(x)$, and rewrite it in the form $dy = f(x)dx$ we can find the antiderivative of both sides using the integration symbol $\int$. That is,

$$y = \int dy = \int f(x)dx = F(x) + C$$

Each piece of this equation has a name that I will refer to: The integrand is $f(x)$, the variable of integration is given by $dx$, the antiderivative of $f(x)$ is $F(x)$, and the constant of integration is $C$. The term indefinite integral is a synonym for antiderivative.
Note: Differentiation and anti-differentiation are “inverse” operations of each other. That is, if you find the antiderivative of a function \( f \), then take the derivative, you will end up back at \( f \). Similarly, if you take the derivative, the antiderivative takes you back.

Some Basic Integration Rules:

\[
\int 0 \, dx = C \quad \int k \, dx = kx + C \quad \int kf(x) \, dx = k \int f(x) \, dx
\]

\[
\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx
\]

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1
\]

Important distinction!

We can also consider all the trig derivatives and go backwards to find their integrals.

Examples: For each function, rewrite then integrate and finally simplify.

1. \[
\int \sqrt[3]{x} \, dx = \int x^{\frac{1}{3}} \, dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4}x^{\frac{4}{3}} + C
\]

\[
\sqrt[3]{x} = x^{\frac{1}{3}}
\]

2. \[
\int \frac{1}{4x^2} \, dx = \int \frac{1}{4} x^{-2} \, dx = \frac{1}{4} \int x^{-2+1} \, dx = \frac{1}{4} \frac{x^{-1}}{-1} + C = \frac{1}{4} \frac{x^{-1}}{-1} + C = \frac{x^{-1}}{-4} + C = -\frac{1}{4x} + C
\]

3. \[
\int \frac{1}{\sqrt{x}} \, dx = \int \sqrt[6]{x} \, dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{3}{5}x^{\frac{5}{3}} + C
\]

4. \[
\int x(x^3 + 1) \, dx = \int x^{4+1} + \frac{x^{11}}{11+1} \, dx = \frac{x^{5+1}}{5+1} + \frac{x^{12}}{12+1} + C = \frac{x^5}{5} + \frac{x^2}{2} + C
\]

No product rule for integrals!
5. \( \int \frac{1}{(3x)^2} \, dx = \int \frac{1}{9x^2} \, dx = \frac{1}{9} \int x^{-2} \, dx = \frac{1}{9} \frac{x^{-2+1}}{-2+1} + C = \frac{1}{9} \frac{x^{-1}}{-1} + C = -\frac{1}{9x} + C \)

On the homework:

6. \( \int \frac{1}{x^{\sqrt{5}}} \, dx = \int x^{-\sqrt{5}/5} \, dx = \frac{x^{-\sqrt{5}/5+1}}{-\sqrt{5}/5+1} + C = \frac{x^{-\sqrt{5}/5}}{-\sqrt{5}/5} + C = -\frac{\sqrt{5}}{5} x^{-\sqrt{5}/5} + C = -\frac{\sqrt{5}}{5} + C \)

Examples: Find the indefinite integral and check the result by differentiation.

1. \( \int (12-x) \, dx = 12x - \frac{x^2}{2} + C \)

Check: \( \frac{d}{dx} \left( 12x - \frac{x^2}{2} + C \right) = 12 - \frac{2x}{2} + 0 = 12 - x \quad \checkmark \)

2. \( \int (8x^3 - 9x^2 + 4) \, dx = \frac{8x^4}{4} - \frac{9x^3}{3} + 4x + C = 2x^4 - 3x^3 + 4x + C \)

Check: \( \frac{d}{dx} (2x^4 - 3x^3 + 4x + C) = 8x^3 - 9x^2 + 4 + 0 = 8x^3 - 9x^2 + 4 \quad \checkmark \)

3. \( \int \left( \sqrt{x} + \frac{1}{2\sqrt{x}} \right) \, dx = \int \left( x^{1/2} + \frac{1}{2} x^{-1/2} \right) \, dx = \frac{x^{1/2+1}}{1/2+1} + \frac{1}{2} \frac{x^{-1/2+1}}{-1/2+1} + C \)

\( \text{Rewrite} \quad \sqrt{x} + \frac{1}{2\sqrt{x}} = x^{1/2} + \frac{1}{2} x^{-1/2} = \frac{x^{3/2}}{3/2} + \frac{1}{2} \frac{x^{1/2}}{1/2} + C = \frac{2}{3} x^{3/2} + x^{1/2} + C \)

Check: \( \frac{d}{dx} \left( \frac{2}{3} x^{3/2} + x^{1/2} + C \right) = \frac{2}{3} \left( \frac{3}{2} x^{1/2} \right) + \frac{1}{2} x^{-1/2} + 0 = \sqrt{x} + \frac{1}{2\sqrt{x}} \quad \checkmark \)
4. \[ \int \frac{x^2 + 2x - 3}{x^4} \, dx = \int \left( x^{-1} + 2x^{-2} - 3x^{-4} \right) \, dx = \frac{x^{-1}}{-1} + 2 \frac{x^{-2}}{-2} - 3 \frac{x^{-3}}{-3} + C = -x^{-1} - x^{-2} + x^{-3} + C \]

No quotient rule for integrals!

\[ \frac{x^2 + 2x - 3}{x^4} = \frac{x^2}{x^y} + 2 \frac{x}{x^4} - 3 \frac{x^{-3}}{x^y} = x^{-1} + 2x^{-3} - 3x^{-y} \]

Check: \[ \frac{d}{dx} (-x^{-1} - x^{-2} + x^{-3} + C) = -(-1x^{-2}) + 2x^{-3} - (-3x^{-4}) + 0 = x^{-2} + 2x^{-3} - 3x^{-4} \]

\[ \sqrt{ \]
Example: Find the equation of \( y \) given \( \frac{dy}{dx} = 2x - 1 \) that has the particular point \((1, 1)\) as part of its solution set.

\[
y = \int dy = \int (2x - 1) \, dx = \frac{2x^2}{2} - x + C = x^2 - x + C
\]

Since \((1, 1)\) is a solution, we can substitute to find a specific value of \(C\):

\[
1 = (1)^2 - 1 + C
\]

\[
1 = 1 - 1 + C
\]

\[
1 = C
\]

\[\text{Solution: } y = x^2 - x + 1\]

Example: Solve the differential equation.

1. \( f'(x) = 6x^2, \quad f(0) = -1 \)

\[
f(x) = \int 6x^2 \, dx = 6 \frac{x^3}{3} + C = 2x^3 + C
\]

\[
-1 = 2(0)^3 + C
\]

\[
-1 = C
\]

\[
f(x) = 2x^3 - 1
\]

2. \( f'(p) = 10p - 12p^3, \quad f(3) = 2 \)

\[
f(p) = \int (10p - 12p^3) \, dp = 10 \frac{p^2}{2} - 12 \frac{p^4}{4} + C = 5p^2 - 3p^4 + C
\]

\[
2 = 5(3)^2 - 3(3)^4 + C
\]

\[
2 = 45 - 243 + C
\]

\[
200 = C
\]

\[
f(p) = 5p^2 - 3p^4 + 200
\]

3. \( h''(x) = \sin x, \quad h'(0) = 1, \quad h(0) = 6 \)

\[
h'(x) = \int h''(x) \, dx = \int \sin x \, dx = -\cos x + C
\]

\[
1 = -\cos(0) + C
\]

\[
1 = -1 + C
\]

\[
2 = C
\]

\[
h'(x) = -\cos x + 2
\]

\[
h(x) = \int (-\cos x + 2) \, dx = -\sin x + 2x + C
\]

\[
l_0 = -\sin(0) + 2(0) + C
\]

\[
l_0 = C
\]
Example: A particle, initially at rest, moves along the x-axis such that its acceleration at time \( t > 0 \) is given by \( a(t) = \cos t \). At the time \( t = 0 \), its position is \( x = 3 \).

a) Find the velocity and position functions for the particle.

b) Find the values of \( t \) for which the particle is at rest.

\[ a(t) = \cos t \]

\[ v(t) = \int a(t) \, dt = \int \cos t \, dt = \sin t + C \quad \text{at rest has } v(0) = 0 \]

\[ v(t) = \sin t \]

\[ x(t) = \int v(t) \, dt = \int \sin t \, dt = -\cos t + C \quad 3 = -\cos(0) + C \]

\[ x(t) = -\cos t + 3 \]

\[ 3 = -1 + C \]

\[ 4 = C \]

b) Particle at rest \( \Rightarrow \) velocity is 0

\[ v(t) = \sin t = 0 \quad \text{when } t = k\pi \text{ for any integer } k \]