4.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTToC) – If a function \( f \) is continuous on the closed interval \([a, b]\) and \( F \) is an antiderivative of \( f \) on the interval \([a, b]\), then
\[
\int_a^b f(x)\,dx = F(b) - F(a).
\]

Examples: Evaluate.

1. \[
\int_4^9 5\,dv = 5\bigg|_4^9 = 5(9) - 5(4) = 45 - 20 = 25
\]
   evaluated from 4 to 9

2. \[
\int_2^5 (-3x + 4)\,dx = \left(-\frac{3x^2}{2} + 4x\right)\bigg|_2^5 = \left(-\frac{3(5)^2}{2} + 4(5)\right) - \left(-\frac{3(2)^2}{2} + 4(2)\right)
\]
   \[= \left(-\frac{75}{2} + 20\right) - \left(-6 + 8\right) = -\frac{77}{2} + 18 = -\frac{39}{2}
\]

3. \[
\int_{-1}^1 (t^3 - 9t)\,dt = \left(\frac{t^4}{4} - \frac{9t^2}{2}\right)\bigg|_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(-\frac{1}{4} + \frac{9}{2}\right) = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(-\frac{1}{4} + \frac{9}{2}\right) = 0
\]
   Note: \( y = t^3 - 9t \) is an odd function over interval \([-1, 1]\) so we expected 0.

4. \[
\int_{-2}^1 \left(u - \frac{1}{u^2}\right)\,du = \left[u - \frac{1}{u}\right]\bigg|_{-2}^1 = \left(\frac{1}{2} - \frac{1}{1}\right) - \left(-\frac{1}{2} - \frac{1}{(-1)^2}\right) = \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{2} - 1\right) = -2
\]

5. \[
\int_0^2 (2-x)\sqrt{x}\,dx = \left(2x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)\bigg|_0^2 = \left(2 \cdot \frac{8}{2} - \frac{2^{\frac{5}{2}}}{\frac{5}{2}}\right) - \left(\frac{4}{3}\sqrt{4} - \frac{2}{3}\sqrt{2^2}\right) = \left(\frac{4}{3}\sqrt{4} - \frac{2}{3}\sqrt{4}\right) - 0
\]
   Still no product rule
   \((2-x)x^{\frac{3}{2}} = 2x^{\frac{5}{2}} - x^{\frac{5}{2}}\)
Examples: Evaluate.

1. \[ \int_0^\pi (2 + \cos x) \, dx = \left( 2x + \sin x \right) \bigg|_0^\pi = (2\pi + \sin \pi) - (2\pi + \sin 0) = 2\pi \]

2. \[ \int_0^{\pi/4} \frac{\sec^2 \theta}{\tan^2 \theta + 1} \, d\theta = \int_0^{\pi/4} \frac{\sec^2 \theta}{\sec^2 \theta} \, d\theta = \int_0^{\pi/4} 1 \, d\theta = \theta \bigg|_0^{\pi/4} = \frac{\pi}{4} - 0 = \frac{\pi}{4} \]

No quotient rule but we do know trig identities and \( \sec^2 \theta = \tan^2 \theta + 1 \)

3. \[ \int_{-\pi/3}^{\pi/3} 4\sec \theta \tan \theta \, d\theta = \left( 4\sec \theta \right) \bigg|_{-\pi/3}^{\pi/3} = 4\sec \left( \frac{\pi}{3} \right) - 4\sec \left( -\frac{\pi}{3} \right) = 4\left( 2 \right) - 4\left( -2 \right) = 8 \]

Examples: Find the area of the region bounded by the graphs of the equations.

1. \( y = 5x^2 + 2, \quad x = 0, \quad x = 2, \quad y = 0 \)
   \[ \text{function limit above } x\text{-axis} \] \[ \int_0^2 (5x^2 + 2) \, dx = \left( \frac{5x^3}{3} + 2x \right) \bigg|_0^2 = \left( \frac{5(2)^3}{3} + 2(2) \right) - \left( \frac{5(0)^3}{3} + 2(0) \right) = \frac{52}{3} \]

exact, not decimal approximation

2. \( y = 1 + \sqrt{x}, \quad x = 0, \quad x = 8, \quad y = 0 \)
   \[ \int_0^8 \left( 1 + \sqrt{x} \right) \, dx = \left( x + \frac{2}{3} x^{3/2} \right) \bigg|_0^8 = \left( 8 + \frac{2}{3} \cdot 2\sqrt{8} \right) - \left( 0 + \frac{2}{3} \cdot 2 \right) = 8 + 12 = 20 \]

The Mean Value Theorem for Integrals – If \( f \) is continuous on the closed interval \([a, b]\), then there exists a number \( c \) in the closed interval \([a, b]\) such that \[ \int_a^b f(x) \, dx = f(c)(b-a) \]
Definition of Average Value on an Interval – If \( f \) is integrable on the closed interval \([a, b]\), then the average value of \( f \) on the interval is

\[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**Note the average value is \( f(c) \) from MVT for Integrals**

Examples: Find the value(s) of \( c \) guaranteed by the MVTfI for the function over the given interval.

1. \( f(x) = x^3 \), \([0,3]\)
   
   \[
   \begin{align*}
   f(c) &= c^3 \\
   b-a &= 3-0 = 3 \\
   \int_0^3 x^3 \, dx &= \left[ \frac{x^4}{4} \right]_0^3 = \frac{3^4}{4} - 0 = \frac{81}{4} \\
   \Rightarrow \quad \frac{81}{4} &= 3c^3 \\
   c &= \sqrt[3]{\frac{27}{4}} = \frac{3}{\sqrt[3]{4}}
   \end{align*}
   \]

2. \( f(x) = x - 2\sqrt{x} \), \([0,2]\)
   
   \[
   \begin{align*}
   f(c) &= c - 2\sqrt{c} \\
   b-a &= 2 \\
   \int_0^2 (x - 2\sqrt{x}) \, dx &= \left[ \frac{x^2}{2} - \frac{4}{3}x^{3/2} \right]_0^2 = \left( \frac{2^2}{2} - \frac{4}{3} \cdot 2^{3/2} \right) - 0 = \frac{(2)^2}{2} - \frac{4}{3} \cdot 2^{3/2} \\
   \Rightarrow \quad 2 - \frac{8}{3} &= (c - 2\sqrt{c})^2 \\
   \Rightarrow \quad 1 - \frac{4\sqrt{c}}{3} &= c - 2\sqrt{c} \\
   \Rightarrow \quad 0 &= c - 2\sqrt{c} - 1 + \frac{4\sqrt{c}}{3} \\
   \Rightarrow \quad c &= \frac{(-6(2\sqrt{c} - 3) \pm 3)^2}{9}
   \end{align*}
   \]

3. \( f(x) = \cos x \), \([-\pi/3, \pi/3]\)
   
   \[
   \begin{align*}
   f(c) &= \cos c \\
   b-a &= \frac{\pi}{3} - (-\frac{\pi}{3}) = \frac{2\pi}{3} \\
   \int_{-\pi/3}^{\pi/3} \cos x \, dx &= \left[ \sin x \right]_{-\pi/3}^{\pi/3} = \sin \left( \frac{\pi}{3} \right) - \sin \left( -\frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} - \left( -\frac{\sqrt{3}}{2} \right) = \sqrt{3} \\
   \Rightarrow \quad \frac{2\pi}{3} \cos c &= \sqrt{3} \\
   \cos c &= \frac{3\sqrt{3}}{2\pi} \\
   \Rightarrow \quad c &= \cos^{-1} \left( \frac{3\sqrt{3}}{2\pi} \right) \\
   \Rightarrow \quad c &\approx 0.597 \text{ rad}
   \end{align*}
   \]

In radians as the interval is given.
Examples: Find the average value of the function over the given interval and all values of $x$ in the interval for which the function equals its average value.

1. $f(x) = 9 - x^2$, $[-3, 3]$

$$\text{Average value} \quad f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$f(c) = \frac{1}{3-(-3)} \int_{-3}^{3} (9-x^2) \, dx = \frac{1}{6} \left[ 9x - \frac{x^3}{3} \right]_{-3}^{3} = \frac{1}{6} \left[ 9(3) - \left( \frac{3^3}{3} \right) \right] - \frac{1}{6} \left[ 9(-3) - \left( \frac{(-3)^3}{3} \right) \right]$$

$$= \frac{1}{6} (27 - 9) - \frac{1}{6} (-27 + 9)$$

$$= \frac{1}{6} (18) - \frac{1}{6} (-18) = 3 + 3 = 6$$

2. $f(x) = x^3$, $[0, 1]$

$$f(c) = \frac{1}{1-0} \int_0^1 x^3 \, dx = \left. \left( \frac{x^4}{4} \right) \right|_0^1 = \frac{1}{4}$$

3. $f(x) = 4x^3 - 3x^2$, $[-1, 2]$

$$f(c) = \frac{1}{2-(-1)} \int_{-1}^{2} (4x^3 - 3x^2) \, dx = \frac{1}{3} \left[ x^4 - x^3 \right]_{-1}^{2} = \frac{1}{3} \left( (2)^4 - (2)^3 \right) - \frac{1}{3} \left( (-1)^4 - (-1)^3 \right)$$

$$= \frac{1}{3} (16 - 8) - \frac{1}{3} (1 + 1)$$

$$= \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$$

Graphically, $x = 1.137$
4. \( f(x) = \sin x, \ [0, \pi] \)

\[
\frac{1}{\pi - 0} \int_{0}^{\pi} \sin x \, dx = \frac{1}{\pi} (-\cos \pi) - \frac{1}{\pi} (-\cos 0) = \frac{1}{\pi} (-(-1)) - \frac{1}{\pi} (-1)
\]

\[
= \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}
\]

\[
\int_{a}^{b} f(c) = \frac{2}{\pi} \quad \text{avg value}
\]

\[
\sin x = \frac{2}{\pi} \quad \text{gives} \quad x = \sin^{-1} \left( \frac{2}{\pi} \right) \approx 0.6901 \text{ rad}
\]

The Second Fundamental Theorem of Calculus – If \( f \) is continuous on an open interval \( I \) containing \( a \), then, for every \( x \) in the interval \[
\frac{d}{dx} \left[ \int_{a}^{x} f(t) \, dt \right] = f(x)
\]

Examples: Use the Second Fundamental Theorem of Calculus to find \( F'(x) \).

1. \( F(x) = \int_{1}^{x} \frac{t^2}{t^2 + 1} \, dt \)

\[
\int_{1}^{e} \text{f}(t) \quad \rightarrow \quad \frac{d}{dx} \int_{1}^{x} \text{f}(t) \, dt = \frac{x^2}{x^2 + 1}
\]

2. \( F(x) = \int_{2}^{x} \frac{1}{t} \, dt \)

\[
\int_{2}^{4} \text{f}(t) = \frac{1}{4} \quad \text{so} \quad \frac{d}{dx} \int_{2}^{x} \text{f}(t) \, dt = \frac{1}{x^3}
\]

However, the upper limit is \( x^2 \) with derivative \( 2x \)

\[
\frac{d}{dx} \int_{2}^{x} \text{f}(t) \, dt = \frac{1}{(x^2)^3}, \quad 2x = \frac{2}{x^5}
\]

3. \( F(x) = \int_{0}^{\sin x} 3t^5 \, dt \)

\[
\int_{0}^{u} \text{f}(t) = 3u^6 \quad \text{so} \quad \frac{d}{dx} \int_{0}^{\sin x} 3t^5 \, dt = 3 \sin^5 x \cos x
\]

\( u = \sin x \quad \text{hence} \quad du = \cos x \)