Counting topologies of metric holomorphic polynomial field with simple zeros

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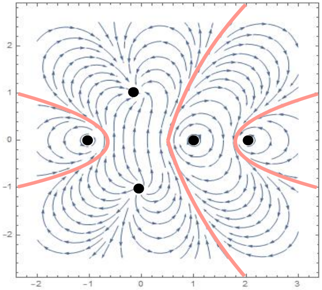
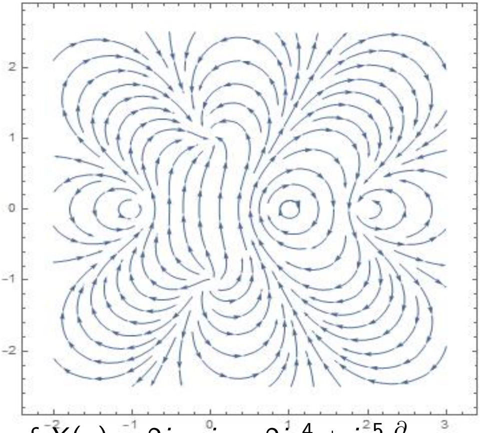
XXXI Semana Nacional de
Investigación y Docencia en Matemáticas
Taller de Estructuras Geométricas y Combinatoria
Universidad de Sonora (online)
May 27, 2021

AD supported by Simons Foundation Grant 516801

Setting the scene: Trees from flow diagrams

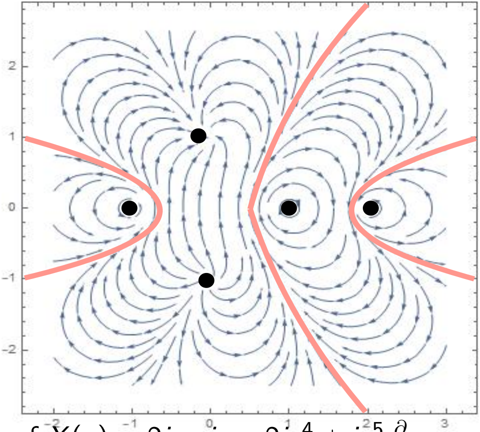


Metric holomorphic polynomial field with simple zeros



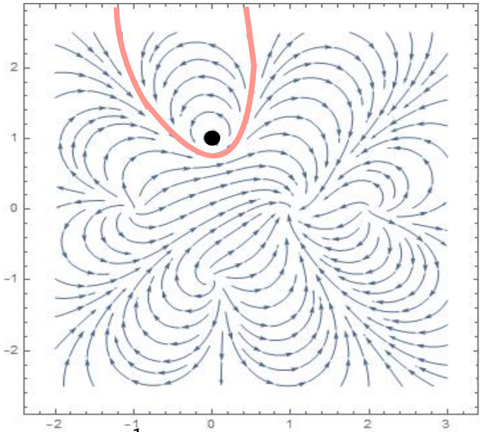
Phase portrait of $X(z) = 2i - iz - 2iz^4 + iz^5 \frac{\partial}{\partial z}$

Metric holomorphic polynomial field with simple zeros



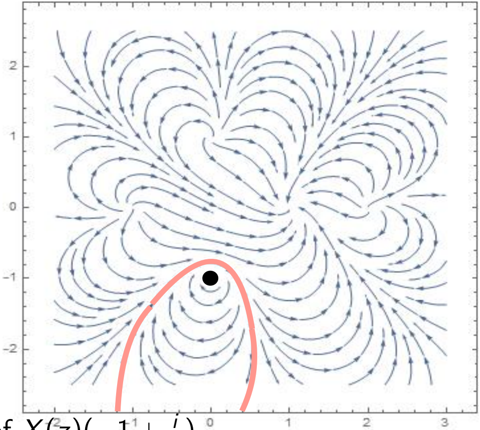
Phase portrait of $X(z) = 2i - iz - 2iz^4 + iz^5 \frac{\partial}{\partial z}$

Complex rotation



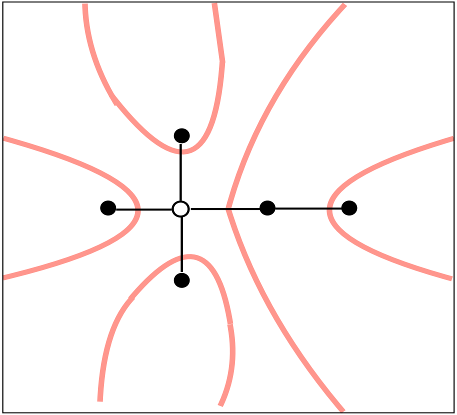
Phase portrait of $X(z)(\frac{1}{2} - i)$

Complex rotation



Phase portrait of $X(z)(-1+\frac{i}{2})$

Put it all together, and get a graph



Trees

So we are looking at unlabeled trees with black and white vertices

- no white vertices are adjacent to each other
- each white vertex is adjacent to at least three black vertices
- no restriction on neighbors of black vertices

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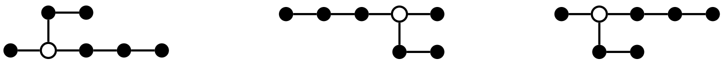
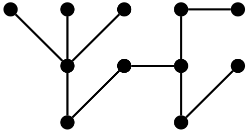
We want to count such trees up to rotation (but not reflection)

Example

The first two are the same, but the third is different.



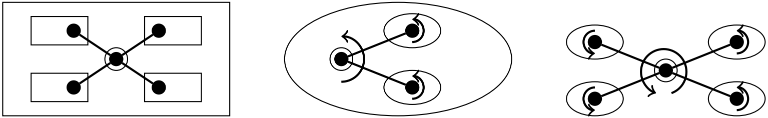
Flashback: Counting (unlabeled) trees



How to grow different kinds of rooted trees, recursively

- Rooted trees:
  - $\mathcal{A} = X \cdot E(\mathcal{A})$,
  - E stands for "set of"
- Ordered rooted tree:
  - $\mathcal{A}_L = X \cdot L(\mathcal{A}_L)$
  - L stands for "linear order"
- Planar rooted trees:
  - $P = X + X \cdot C(\mathcal{A}_L)$
  - C stands for "cyclic order"

Example



Unrooting I: Center of tree

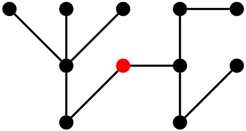
Definition

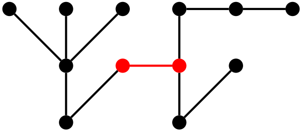
Center of a tree is the set of vertices v that minimize

$$\max_u d(u, v)$$

It is always either a single vertex, or an edge.

Example





Unrooting I: Center of tree

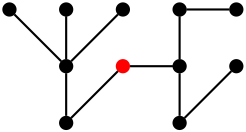
Definition

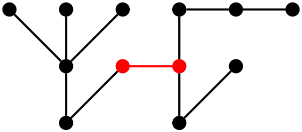
Center of a tree is the set of vertices v that minimize

$$\max_u \mathrm{d}(u, v)$$

It is always either a single vertex, or an edge. So this naturally roots a tree at either a vertex or an edge.

Example





Unrooting II: Dissymmetry theorem

Theorem (Dissymmetry)

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

where $\mathfrak{a}$ *denotes unrooted trees and* E_2 *is the species of sets with exactly two elements.*

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Proof.

(Sketch) LHS is trees rooted at a vertex or an edge. RHS is trees (unrooted) or ordered pair of rooted trees.

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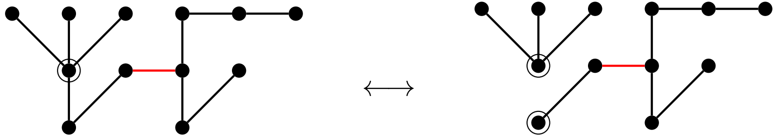
Proof.

(Sketch) LHS is trees rooted at a vertex or an edge. RHS is trees (unrooted) or ordered pair of rooted trees. So we need isomorphism between trees rooted at vertex or edge other than the center, with ordered pairs of rooted trees. □



Quick note about unlabeling

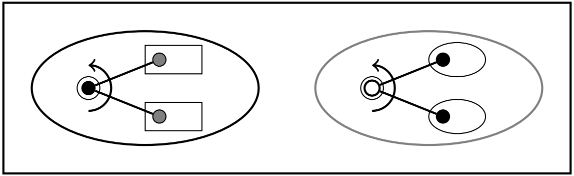
Theorem (Dissymmetry)

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

where $\mathfrak{a}$ denotes unrooted trees and E_2 is the species of sets with exactly two elements.

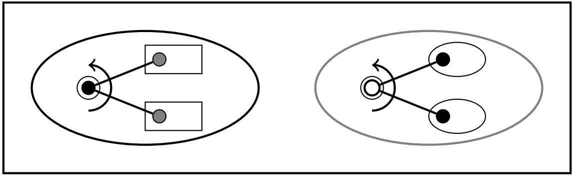
Dissymmetry theorem allows us to count unrooted, but still labeled trees. To unlabel the trees, we need "cycle index series".

Return to the present day: Counting our trees



Black and white vertices, not at the root

Similar to ordered rooted trees, but now color-aware



$$Y_1 = X_1 \cdot L(Y_1 + Y_2) \qquad Y_2 = X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_1 + Y_2))$$

$$Y_3 = Y_1 + Y_2 = X_1 \cdot L(Y_3) + X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_3))$$

Recursive equation

$$Y_3 = Y_1 + Y_2 = X_1 \cdot L(Y_3) + X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_3))$$

$$y_3 = x_1 \ell + x_2 \frac{(x_1 \ell)^2}{1 - (x_1 \ell)}$$

where $\ell = \frac{1}{1 - y_3}$.

Recursive equation

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where $\ell = \frac{1}{1-y_3}$. Simplifying,

$$x_1 + x_1^2(x_2 - 1) - (y_3 - 1)^2 y_3 - x_1 y_3^2 = 0.$$

Unique real root $y_3(x_1, x_2) =$

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Unique real root $y_3(x_1, x_2) =$

$$\frac{\frac{2-x1}{3} + \left(2^{1/3}\left(-1+4x1-x1^2\right)\right)}{\left(3\left(2-12x1+15x1^2+2x1^3-27x1^2x2+\sqrt{\left(4(-1+4x1-x1^2)^3+(2-12x1+15x1^2+2x1^3-27x1^2x2)^2\right)}\right)^{1/3}\right)}$$

$$-\frac{1}{3\ 2^{1/3}}\left(2 - 12x1 + 15x1^2 + 2x1^3 - 27x1^2x2 + \sqrt{\left(4\left(-1+4x1-x1^2\right)^3 + \left(2 - 12x1 + 15x1^2 + 2x1^3 - 27x1^2x2\right)^2\right)}\right.$$

Dissymmetry again

Recall

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

The same arguments apply. But now, paying attention to color,

$$\mathcal{A}_R = (X_1 \cdot (1 + C(Y_3))) + (X_2 \cdot C_{\geq 3}(X_1 \cdot L(Y_3)))$$

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$$E_2(\mathcal{A}_R) = E_2(Y_1) + Y_2 \cdot Y_1 = E_2(Y_3) - E_2(Y_2)$$

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\mathcal{A}_R^2 &= Y_1^2 + 2Y_1Y_2 = Y_3^2 - Y_2^2
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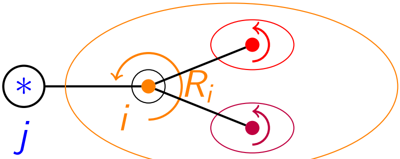
\end{aligned}$$

And then, to remove labels, again bring in cycle index series.

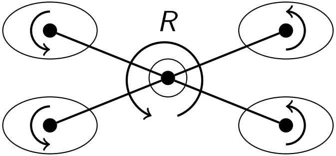
Adding color



How to grow a general variety of rooted trees, recursively

- R-enriched rooted trees: $\mathfrak{a}_R^\bullet = \mathcal{A}_R = X \cdot R(\mathcal{A}_{R'})$

Example



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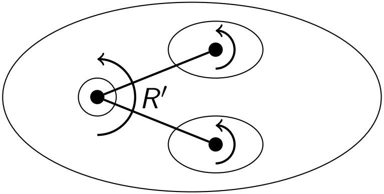

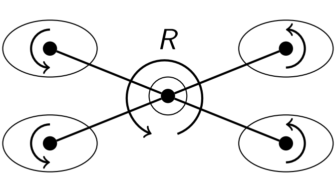

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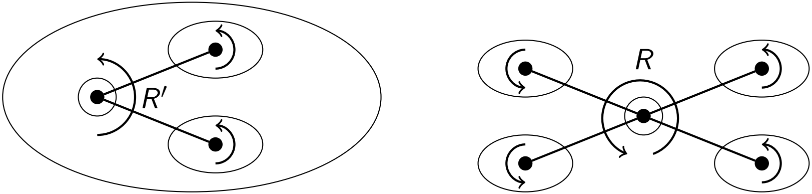
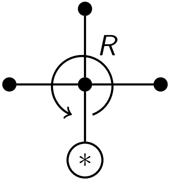
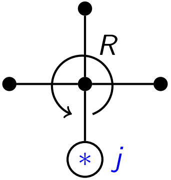


Definition (derivative R')



Adding color

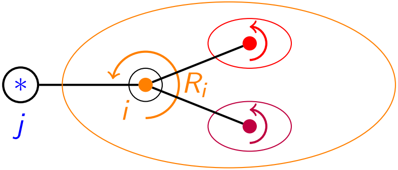
$$R^{(j)} = \frac{\partial R}{\partial X_j}$$

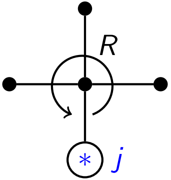


Adding color

$$R^{(j)} = \frac{\partial R}{\partial X_j}$$

$$(\mathcal{A}_i)_{R^{(j)}} = X_i R_i^{(j)}(((\mathcal{A}_1)_{R^{(i)}}, (\mathcal{A}_2)_{R^{(i)}}, \ldots, (\mathcal{A}_k)_{R^{(i)}}))$$





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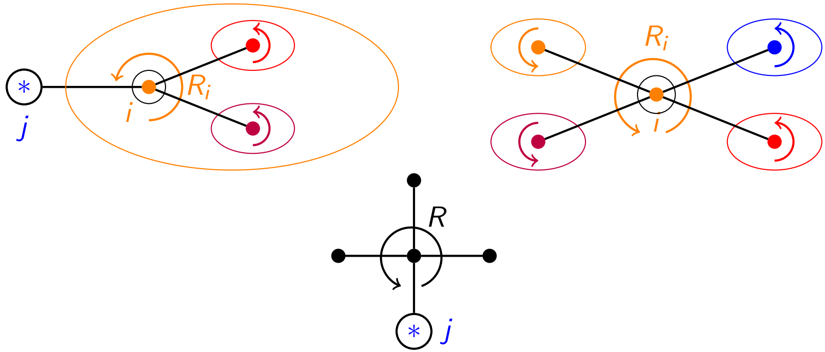
$$\mathfrak{a}_R^{\bullet i} = X_i R_i(((\mathcal{A}_1)_{R^{(i)}}, (\mathcal{A}_2)_{R^{(i)}}, \dots, (\mathcal{A}_k)_{R^{(i)}}))$$



Example: Vector fields

$$
\begin{aligned}
R_1(U_1, U_2) &= C(U_1 + U_2) \\
R_1^{(1)}(U_1, U_2) &= R_1^{(2)}(U_1, U_2) = L(U_1 + U_2) \\
R_2(U_1, U_2) &= C_{\geq 3}(U_1) \\
R_2^{(1)}(U_1, U_2) &= L_{\geq 2}(U_1) \\
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$$Y_1 = Y_{1,1} = Y_{1,2} = X_1 R_1^{(1)}(Y_{1,1}, Y_{2,1}) = X_1 L(Y_{1,1} + Y_{2,1})$$

$$Y_2 = Y_{2,1} = X_2 R_2^{(1)}(Y_{1,2}, Y_{2,2}) = X_2 L_{\geq 2}(Y_{1,2})$$

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$$= X_1 L(Y_1 + Y_2)$$
$$Y_2 = Y_{2,1} = X_2 R_2^{(1)}(Y_{1,2}, Y_{2,2}) = X_2 L_{\geq 2}(Y_{1,2})$$
$$= X_2 L_{\geq 2}(Y_1)$$

New example

No restriction on black vertices; white vertices have at most one neighbor of each color.

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R_2(U_1, U_2) &= (1 + U_1)(1 + U_2) \\
R_2^{(1)}(U_1, U_2) &= (1 + U_2) \\
R_2^{(2)}(U_1, U_2) &= (1 + U_1)
\end{aligned}
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$$Y_{2,1} = X_2 R_2^{(1)}(Y_{1,2}, Y_{2,2}) = X_2(1 + Y_{2,2})$$

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$$Y_{2,2} = X_2 R_2^{(2)}(Y_{1,2}, Y_{2,2}) = X_2(1 + Y_1)$$

From this we get the recursive equation for Y_1:

$$Y_1 = X_1 L(Y_1 + Y_{2,1}) = X_1 L(Y_1 + X_2 + X_2^2 + X_2^2 Y_1)$$

Dissymmetry, again

Theorem (D.,F.-A.)

$$\mathfrak{a}_R = \sum_i \left(\mathfrak{a}_R^{\bullet i} + E_2((\mathcal{A}_i)_{R^{(i)}}) - ((\mathcal{A}_i)_{R^{(i)}})^2\right) - \sum_{i<j}(\mathcal{A}_i)_{R^{(j)}} \cdot (\mathcal{A}_j)_{R^{(i)}}$$

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Example (Vector fields)

$$\mathfrak{a}_R = Y_4 + E_2(Y_1) - Y_1^2 - Y_1 \cdot Y_2 = Y_4 + E_2(Y_1) - Y_1 Y_3$$

Dissymmetry, again

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$\mathfrak{a}_R = \sum_i \left(\mathfrak{a}_R^{\bullet i} + E_2((\mathcal{A}_i)_{R^{(i)}}) - ((\mathcal{A}_i)_{R^{(i)}})^2\right) - \sum_{i<j} (\mathcal{A}_i)_{R^{(j)}} \cdot (\mathcal{A}_j)_{R^{(i)}}$

Example (Vector fields)

$\mathfrak{a}_R = Y_4 + E_2(Y_1) - Y_1^2 - Y_1 \cdot Y_2 = Y_4 + E_2(Y_1) - Y_1 Y_3$

Proof.

(Sketch) Mostly same as before.

Dissymmetry, again

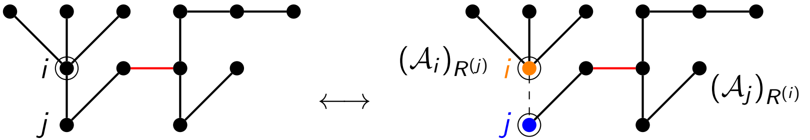
Theorem (D.,F.-A.)

$$\mathfrak{a}_R = \sum_i \left(\mathfrak{a}_R^{\bullet i} + E_2((\mathcal{A}_i)_{R^{(i)}}) - ((\mathcal{A}_i)_{R^{(i)}})^2\right) - \sum_{i<j} (\mathcal{A}_i)_{R^{(j)}} \cdot (\mathcal{A}_j)_{R^{(i)}}$$

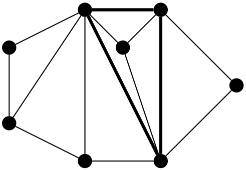
Example (Vector fields)

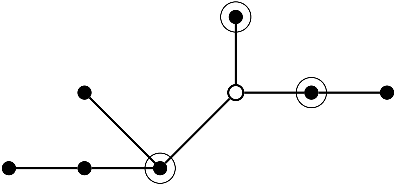
$$\mathfrak{a}_R = Y_4 + E_2(Y_1) - Y_1^2 - Y_1 \cdot Y_2 = Y_4 + E_2(Y_1) - Y_1 Y_3$$

Proof.
(Sketch) Mostly same as before. But the isomorphism between trees rooted at vertex or edge other than the center, with ordered pairs of rooted trees, needs to take into account color. □



Aftermath: Data and Specializations





Data

	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	2	3	6	14	34	95	280	854	2694
1	0	0	1	2	5	16	48	164	559	1952	6872	24520
2	0	0	0	0	1	5	30	146	693	3108	13608	58200
3	0	0	0	0	0	0	2	20	175	1254	7752	44112
4	0	0	0	0	0	0	0	0	7	95	1125	10108
5	0	0	0	0	0	0	0	0	0	0	19	480
6	0	0	0	0	0	0	0	0	0	0	0	0

Data

	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	2	3	6	14	34	95	280	854	2694
1	0	0	1	2	5	16	48	164	559	1952	6872	24520
2	0	0	0	0	1	5	30	146	693	3108	13608	58200
3	0	0	0	0	0	0	2	20	175	1254	7752	44112
4	0	0	0	0	0	0	0	0	7	95	1125	10108
5	0	0	0	0	0	0	0	0	0	0	19	480
6	0	0	0	0	0	0	0	0	0	0	0	0

No white vertices:

Unlabeled plane trees.

Data

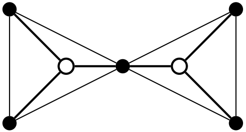
	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	2	3	6	14	34	95	280	854	2694
1	0	0	1	2	5	16	48	164	559	1952	6872	24520
2	0	0	0	0	1	5	30	146	693	3108	13608	58200
3	0	0	0	0	0	0	2	20	175	1254	7752	44112
4	0	0	0	0	0	0	0	0	7	95	1125	10108
5	0	0	0	0	0	0	0	0	0	0	19	480
6	0	0	0	0	0	0	0	0	0	0	0	0

No white vertices:

Unlabeled plane trees.

Minimal black vertices:

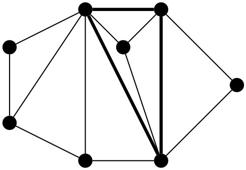
Unlabeled 3-gonal cacti with n triangles.
(Bóna, Bousquet, Labelle, Leroux, 2000)

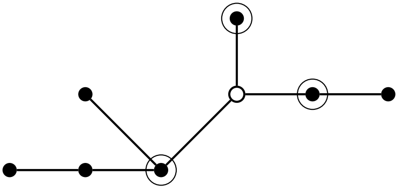


One white vertex

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	1	2	5	16	48	164	559	1952	6872	24520

Triangulations of an *n*-gon with exactly one internal vertex.
(Brown, 1964)

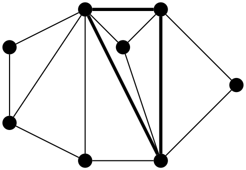


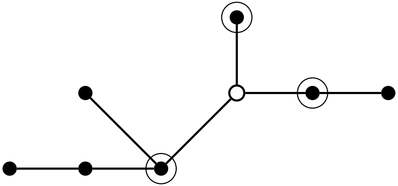


One white vertex

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	1	2	5	16	48	164	559	1952	6872	24520

Triangulations of an *n*-gon with exactly one internal vertex.
(Brown, 1964)





Both are circular orders of (at least three) Catalan-things (ordered rooted trees or rooted triangulations).