What is 0^0, and who decides, and why does it matter?: The role of definitions in mathematics

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- "That one slide justified having you on the project"

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- This was a choice, to make a theorem nicer to state

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- Example that there is some choice in which conditions to include in a definition.

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- No cancellation: $\infty + 5 = \infty + 3$, but $5 \neq 3$.

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  - $\sqrt[3]{-64}$ exists, but $\sqrt{-64}$ does not

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- $60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$
- $60 = 1 \times 1 \times \cdots \times 1 \times 2 \times 2 \times 3 \times 5$

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- (In polynomial factorization, all the non-zero numbers are units)

Empty set

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- False hypothesis makes implication true!

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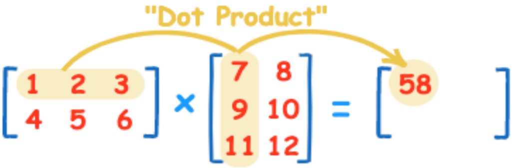
- Yes, so for instance the sum of two polynomials is always a polynomial

Matrix multiplication

- Why is matrix multiplication defined the way it is?



Matrix multiplication

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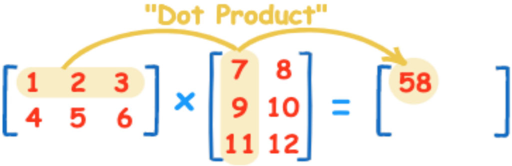

Matrix multiplication

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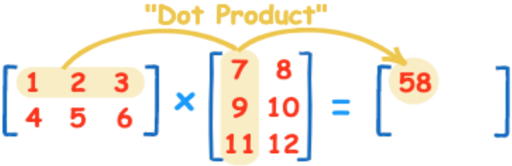
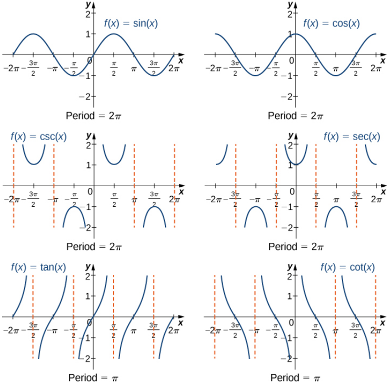
- So the equation comes first, then the definition, not the other way around!



Range of inverse trig functions

- Trig functions are not 1-1



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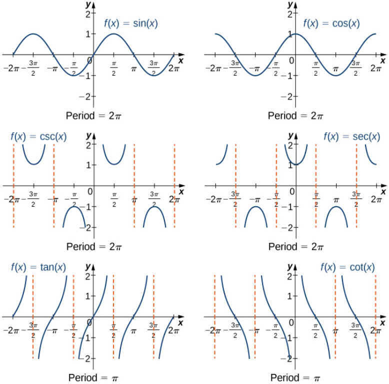

Range of inverse trig functions

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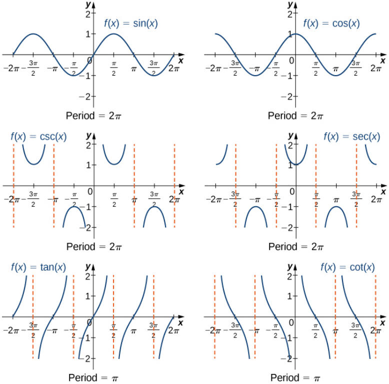
- Which domain do we pick?



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- We should not hide all this from students!